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Self-dual Einstein Hermitian Surfaces

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§1. Introduction

N. Hitchin [4] has proved that a 4-dimensional compact half conformally flat Einstein space of positive scalar curvature is isometric to a 4-dimensional sphere or a complex projective surface with the respective standard metric.

A 4-dimensional almost Hermitian manifold $M = (M, J, g)$ with integrable almost complex structure J is called a Hermitian surface. In the present paper, concerning the above result by Hitchin, we shall prove the following

Theorem A. *Let $M = (M, J, g)$ be a compact self-dual Einstein Hermitian surface. Then M is a Kähler surface of constant holomorphic sectional curvature, i.e., M is one of the following*
(1) *flat,*
(2) *$P^2(\mathbf{C})$ with its standard Fubini-Study metric and*
(3) *a compact quotient of unit disk D^2 with the Bergman metric.*

Remark. C.P. Boyer [2] has asserted the above result without detailed proof. In the present paper, we shall give another explicit proof.

In the sequel, unless otherwise stated, we assume the manifold under consideration to be connected.

§2. Preliminaries

Let $M = (M, J, g)$ be a Hermitian surface and Ω the Kähler form of M given by $\Omega(X, Y) = g(X, JY)$, $X, Y \in \mathfrak{X}(M)$. ($\mathfrak{X}(M)$ denotes the Lie algebra of all differentiable vector fields on M). We assume that M is oriented by the volume form $dM = \frac{1}{2}\Omega^2$. We have

$$(2.1) \qquad d\Omega = \omega \wedge \Omega, \quad \omega = \delta\Omega \circ J.$$

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