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On Symmetry Groups of the MIC-Kepler Problem and Their Unitary Irreducible Representations

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It is well known that the quantized Kepler problem (i.e., the hydrogen atom) admits the symmetry groups, $SO(4)$, $E(3)$ (the Euclidean motion group), or $SO^+(1,3)$ (the proper Lorentz group), according as the energy is negative, zero, or positive (cf. [B-I]). The symmetry groups here stand for Lie groups which act unitarily irreducibly on the Hilbert spaces associated with the energy-spectrum for the Kepler problem. However, only a part of the unitary irreducible representations are realized as the symmetry group for the Kepler problem. A question now arises: Are the other unitary irreducible representations realizable as symmetry groups for a "modified" Kepler problem ?

This question is worked out in this article. Both in classical and quantum mechanics, the Kepler problem is generalized to the MIC-Kepler problem, the Kepler problem along with a centrifugal potential and Dirac's monopole field, which is named after McIntosh and Cisneros [MI-C]. It will be shown that the quantized MIC-Kepler problem exhausts almost all the unitary irreducible representations of $SU(2) \times SU(2)$, $\mathbf{R}^3 \ltimes SU(2)$, or $SL(2,\mathbf{C})$ as the symmetry group, according as the energy is negative, zero, or positive, which groups are the double covers of $SO(4)$, $E(3)$, and $SO^+(1,3)$, respectively. For $SL(2,\mathbf{C})$, the principal series representations are all realizable, but not the others.

§1. Setting up the quantized MIC-Kepler problem

The MIC-Kepler problem is to be defined as a reduced system of the conformal Kepler problem. Consider the principal $U(1)$ bundle $\pi : \mathbf{R}^4 - \{0\} \to \mathbf{R}^3 - \{0\}$ whose projection π and $U(1)$ action Φ_t are given, respectively, by

$$(1.1) \qquad \pi(q) = (2(q_1q_3 + q_2q_4),\ 2(-q_1q_4 + q_2q_3),\ q_1^2 + q_2^2 - q_3^2 - q_4^2),$$

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