Advanced Studies in Pure Mathematics 22, 1993
Progress in Differential Geometry
pp. 1–11

Compactification of Submanifolds in Euclidean Space by the Inversion

Kazuyuki Enomoto

Dedicated to Professor Tominosuke Otsuki

Introduction

Let $\bar{M}$ be an n-dimensional compact connected C^2 submanifold in the N-dimensional Euclidean space R^N. Let Ψ be the inversion of R^N, which is defined by $\Psi(x) = x/|x|^2$ for x in $R^N \cup \{\infty\}$. If the origin O is contained in $\bar{M}$, $\Psi(\bar{M})$ becomes a noncompact, complete, connected C^2 submanifold properly immersed into R^N. If we denote the second fundamental form of $\Psi(\bar{M})$ by B, $|x|^2|B|$ $(x \in \Psi(\bar{M}))$ is bounded on $\Psi(\bar{M})$. In this paper we study the image by the inversion of a noncompact, complete, connected C^2 submanifold M of dimension $n \geq 2$ which is properly immersed into R^N. We are particularly interested in the smoothness of $\Psi(M)$ at the origin O. We say that M satisfies the condition $\mathrm{P}(\alpha)$ if $|x|^\alpha|B|$ $(x \in M)$ is bounded on M. We prove that if M satisfies $\mathrm{P}(2+\varepsilon)$ for some positive constant ε, then the image of each end of M by Ψ is C^2 at O (Theorem 2). Boundedness of $|x|^2|B|$ (i.e., $\mathrm{P}(2)$) is not sufficient to assure that $\Psi(M)$ is C^2 at O, while $\Psi(M)$ is C^1 at O if $\mathrm{P}(1+\varepsilon)$ is satisfied for some $\varepsilon > 0$ (Theorem 1).

Noncompact submanifolds satisfying $\mathrm{P}(1+\varepsilon)$ are studied by Kasue and Sugahara ([4], [5]). They show that those submanifolds become totally geodesic under certain additional conditions on the mean curvature or the sectional curvature. We will make use of some of their results in our proof. As a direct consequence of our theorems, we see that if M satisfies $\mathrm{P}(1+\varepsilon)$, the Gauss map is continuous at infinity, and if M satisfies $\mathrm{P}(2+\varepsilon)$, then M is conformally equivalent to a compact C^2 Riemannian manifold punctured at a finite number of points. We also show that the total integral of the Lipschitz-Killing curvature over the unit normal bundle is an integer if M satisfies $\mathrm{P}(2+\varepsilon)$ (Theorem 3).

Received March 13, 1991.