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Some Observations concerning the Distribution of the Zeros of the Zeta Functions (I)

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§1. Introduction

Let $Z(s)$ be a zeta function which has nice properties like the Riemann zeta function $\zeta(s)$. Let σ_0 be the critical point of $Z(s)$ and suppose that the Riemann Hypothesis (R.H.) holds for $Z(s)$, namely, all the non-trivial zeros of $Z(s)$ are of the form $\sigma_0 + i\gamma$ with a real number γ. The purpose of the present article is to find some of the characteristic properties of the distribution of the zeros of $\zeta(s)$. We shall approach this problem by comparing it with the distribution of the zeros of $Z(s)$ from the following three points of views.

(A) To study the pair correlation of the zeros of $Z(s)$. Namely, to find an asymptotic law for the quantity

$$\sum_{\substack{0<\gamma,\gamma'\leq T \\ 0<\gamma-\gamma'\leq \frac{2\pi\alpha}{\log\frac{T}{2\pi}}}} \cdot 1 \qquad \text{as} \quad T\to\infty,$$

where γ and γ' run over the imaginary parts of the non-trivial zeros of $Z(s)$ and α is a positive number.

(B) To find an asymptotic law for the mean value

$$\int_0^T \left(S_Z(t+\Delta) - S_Z(t)\right)^2 dt \qquad \text{as} \quad T\to\infty,$$

where $S_Z(t) = \frac{1}{\pi}\arg Z(\sigma_0 + it)$ as usual and $\Delta > 0$.

(C) To find an asymptotic law for the sum

$$\sum_{0<\gamma\leq T} \left|Z(\sigma_0 + i(\gamma+\Delta))\right|^2 \qquad \text{as} \quad T\to\infty.$$

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