Advanced Studies in Pure Mathematics 21, 1992
Zeta Functions in Geometry
pp. 219–226

Multiple Zeta Functions: An Example

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We try to construct multiple zeta functions as follows. Let $Z_i(s)$ be usual zeta functions for $i = 1, \ldots, r$: we assume that they are defined by Euler products and meromorphic on $\mathbf{C}$ of finite order with functional equations. Let $m_i : \mathbf{C} \longrightarrow \mathbf{Z}$ denote the multiplicity of zeros and poles of $Z_i(s)$ so that we have the Hadamard expression

$$Z_i(s) = \prod_{\rho \in \mathbf{C}} (s - \rho)^{m_i(\rho)}$$

up to a factor $\exp(P(s))$ for a polynomial $P(s)$. Here we simplify the notation by omitting the usual exponential factor making the convergence, since at the first level we are mainly interested in zeros and poles of (multiple) zeta functions. For more precise studies, it is better to consider such a product via zeta regularized determinant (cf. [2a]). Now we define a "multiple zeta function" by

$$Z_1(s) \otimes \cdots \otimes Z_r(s) = \prod_{\rho_i \in \mathbf{C}} (s - (\rho_1 + \cdots + \rho_r))^{m(\rho_1, \ldots, \rho_r)}$$

where

$$m(\rho_1, \ldots, \rho_r) = m_1(\rho_1) \cdots m_r(\rho_r) \times \begin{cases} 1 & \text{if all } \ \Im(\rho_i) \geq 0, \\ (-1)^{r-1} & \text{if all } \ \Im(\rho_i) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to see that $Z_1(s) \otimes \cdots \otimes Z_r(s)$ has an Euler product expression at least formally, but we must remark that the above "parity condition" is crucial to have a neat Euler product (cf. [2b, p.336]). As noted in [2b] and [2c] these multiple zeta functions would be considered to be associated to multiple categories of Ehresmann (or higher stacks

Received October 20, 1990.