Advanced Studies in Pure Mathematics 21, 1992
Zeta Functions in Geometry
pp. 127–140

Some Exact Trace Formulæ

Charles L. Epstein

§1. Introduction

In the study of the singularities of the trace of the wave operator on Riemannian manifolds there is a dichotomy between the constant curvature case and the general case. On an $(n+1)$-dimensional manifold of constant negative curvature, M one studies

$$\operatorname{tr}(\exp t(\Delta + \left(\frac{n}{2}\right)^2)^{\frac{1}{2}}). \tag{1.1}$$

In this case the Selberg trace formula provides an exact expression in terms of the lengths of the closed geodesics on M. In the general case one considers

$$\operatorname{tr}(\exp t(\Delta)^{\frac{1}{2}}). \tag{1.2}$$

As such, flat tori are the only examples where the trace in (1.2) is computed explicitly. In fact, these are the only examples where precise bounds for the error term in the Guillemin-Duistermaat formula are known. In this note we narrow the gap, a bit, by showing that there is a general procedure for obtaining an exact formula for

$$\operatorname{tr}(\exp t(L + \alpha^2))$$

whenever one has an exact formula for

$$\operatorname{tr}(\exp t(L)^{\frac{1}{2}}).$$

Here L can be taken as a shifted Laplace operator but the idea applies with much greater generality. As a special case we obtain the trace in (1.2) for the Laplace operator of a compact hyperbolic manifold. Our calculations serve, at least in this special case, to clarify the nature of the error term in the Guillemin-Duistermaat formula.

Received October 30, 1990.

Research supported in part by The Sloan Foundation, by NSF Grant DMS-901957, and by SFB 170, Göttingen