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The Relation between the η-Invariant and the Spin Representation in Terms of the Selberg Zeta Function

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§1. Introduction

Roughly speaking, we will show that the "ratio"

$$\frac{\det(\text{Cayley transform of } \sqrt{\Delta})}{\text{SZF attached to the difference of spin representations}}$$

is essentially equal to the exponential of η-invariant. Of course, we mean "det" the functional determinant. So we need the discussion on the regularization of these kind of determinants. This invariant was introduced by Atiyah-Patodi-Singer and indices the spectral asymmetry of (an odd dimensional) Riemannian manifold. Namely, at least formally (or symbolically),

$$\begin{aligned}
&\eta\text{-invariant}\\
=&\text{the internal index (signature) of an infinite quadratic form}\\
&\overline{{}^t d\mathbb{X}}\sqrt{\Delta}d\mathbb{X} \text{ on } d\Omega^{(\text{dimension } -1)/2}\\
=&\sharp\{\text{positive eigenvalues of } \sqrt{\Delta}\} - \sharp\{\text{negative eigenvalues of } \sqrt{\Delta}\}.
\end{aligned}$$

By the way, as to the result on the SZF related to the spin representations on the compact quotient of the hyperbolic space of dimension $4n-1$, there is a work by Milson. For computing the η-invariant in terms of the SZF, he found the intermediate formula (Selberg trace formula for "odd" type). Our result mentioned above is the one for the Milson's type SZF. (But we also consider the objects which are associated with the finite dimensional unitary representations of the fundamental group.) However this type of zeta function is actually defined by the "difference" of the two spin representations that decompose the action of $SO(2\rho)$ on

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