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Einstein Metrics in Complex Geometry: An Introduction

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Based on Calabi's pioneering work, systematic studies of Einstein-Kähler metrics commenced in mid-seventies with the affirmative solution of Calabi's conjecture by Aubin [2] and Yau [23]. Nowadays, via efforts of Yau himself and also other mathematicians, such metrics are shown to play by far important roles not only in differential geometry but also in algebraic geometry. Moreover, in eighties, the concept of Einstein-Hermitian metrics was introduced by S. Kobayashi (cf. [14]), and the resulting analogy of Calabi's conjecture to vector bundle cases, known as Hitchin-Kobayashi's conjecture, was solved affirmatively by Donaldson [9], [10], Uhlenbeck and Yau [22]. This now allows us to make differential-geometric studies of moduli spaces of vector bundles.

In view of these facts, we shall devote Volume 18-II to surveys of recent progress on the study of these metrics. By organizing three working groups, we divided the whole subjects into three categories:

(1) "Einstein-Kähler metrics with positive Ricci curvature" by Futaki, Mabuchi and Sakane.
(2) "Einstein-Kähler metrics with non-positive Ricci curvature" by Bando, Enoki, R. Kobayashi and Sugiyama.
(3) "Yang-Mills connections and Einstein-Hermitian metrics" by Itoh and Nakajima.

All of these are intended to be highly of expository nature but I believe that some particular places therein are written as original works. Now, to provide an introduction to subsequent surveys, we shall briefly discuss basic facts on Einstein metrics in complex geometry.

§1. Einstein-Kähler metrics and Calabi's Conjecture

Let X be an n-dimensional compact connected complex manifold endowed with a Kähler form ω. Then the corresponding Ricci form

$$\mathrm{Ric}(\omega) = \sqrt{-1}\,\bar{\partial}\partial \log \omega^n$$