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Self-Duality of ALE Ricci-Flat 4-Manifolds and Positive Mass Theorem

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Introduction

In [An, Na, BKN] we have developed the theory of the Hausdorff convergence of Einstein 4-manifolds. Then the importance of ALE Ricci-flat 4-manifolds has appealed to us. The phenomenon is very similar to the case of Yang-Mills connections. When the curvatures of Einstein metrics concentrate around a point, an ALE Ricci-flat 4-manifold bubbles off from there (see §2). This corresponds to the phenomenon that a Yang-Mills connection on S^4 bubbles off from the points at which curvatures of Yang-Mills connections concentrate [Uh]. The classification of Yang-Mills connections on S^4 [AHS, Do2] has been essential in applications of the gauge theory to the differential topology ([Do1, FS]), so that it is very plausible that we think the classification of ALE Ricci-flat 4-manifolds is also important. In view of Kronheimer's classification of such spaces under the additional assumption that the spaces are hyperkähler ([Kr]) (see §1), we pose the following question. *Are there Ricci-flat ALE 4-manifolds other than quotients of hyperkähler space ?*

This question is related to physics. For ALE manifolds we can define the *mass* (see §3). If the space is Ricci-flat, the mass vanishes. Some physicists conjecture that the space is self-dual in this case [EGH]. This is a part of *the generalized positive action conjecture*. Although counter-examples of this conjecture are recently constructed by LeBrun [Le], we can show that the conjecture holds under certain topological assumptions (Theorem 3.3).

In this paper we shall make a survey of results on ALE Ricci-flat 4-manifolds and give a partial answer to the above question. In particular we shall prove that a spin Ricci-flat ALE 4-manifold with $\Gamma \subset SU(2)$ has a hyperkähler structure (Theorem 3.3). We shall give some other

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