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An Algebraic Character associated with the Poisson Brackets

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Dedicated to Professor Akio Hattori on his sixtieth birthday

§0. Introduction

Let N be a connected compact Kähler manifold, and $\mathrm{Aut}(N)$ the group of holomorphic automorphisms of N. Then if $c_1(N)_{\mathbb{R}} < 0$ or $c_1(N)_{\mathbb{R}} = 0$, the celebrated solution of Calabi's conjecture by Aubin [2] and Yau [19] asserts that N always admits an Einstein-Kähler metric. In the case $c_1(N)_{\mathbb{R}} > 0$, however, the existence problem is still open, and moreover a couple of obstructions to the existence are known. For instance, Futaki [7] introduced a complex Lie algebra homomorphism $F_N\colon H^0(N, \mathcal{O}(TN)) \to \mathbb{C}$ such that

(1) $F_N = 0$ if N admits an Einstein-Kähler metric;
(2) $F_N \neq 0$ for N in a fairly large family of compact Kähler manifolds (see also Koiso and Sakane [15]).

The purpose of this note is to give a systematic study of the obstruction F_N from a viewpoint of symplectic geometry. For instance, we relate it to the theorem of stationary phase of Duistermaat and Heckman [4], [5]. Another key to our approach is the following (cf. §6):

Theorem 0.1. *For any unipotent subgroup of* $\mathrm{Aut}(N)$*, the corresponding nilpotent Lie subalgebra of* $H^0(N, \mathcal{O}(TN))$ *sits in the kernel of* F_N*. Hence, if* $F_N \neq 0$*, then* N *admits a nontrivial biregular action of the algebraic group* $\mathbb{G}_m$ *(*$= \mathbb{C}^*$ *as a complex Lie group).*

Recall in particular that this theorem implies the identity

(0.2) $$\psi(g) = |\det \phi(g)|^{\gamma} \qquad \text{for all } g \in \mathrm{Aut}(N),$$

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