Advanced Studies in Pure Mathematics 18-I, 1990
Recent Topics in Differential and Analytic Geometry
pp. 271–281

Poincaré Bundle and Chern Classes

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§1. Theorems

Let (X, g) be a compact Kähler surface and P an $SU(2)$ bundle over X of index $k = c_2(P \times_\rho \mathbb{C}^2)$. We denote by $M_k = M_{k,X}$ the set of all gauge equivalence classes of anti-self-dual connections on P.

It is known that (i) the moduli space M_k is a Kähler manifold (possibly with singularities) and the dimension of the non-singular part of M_k is from the Atiyah-Singer index theorem $4k - 3(1 - q(X) + p_g(X))$ ([12],[14]) and (ii) in particular when (X, g) is Ricci flat Kähler, i.e., hyperkähler, the non-singular part $\widehat{M_k}$ is also hyperkähler ([13]).

So, we have

Theorem 1. *The first Chern class $c_1(\widehat{M_k})$ vanishes provided X is hyperkähler.*

This fact was shown also by S. Kobayashi by using complex symplectic geometry ([17]). See also [19].

This theorem shows that the moduli space of anti-self-dual connections inherits the Ricci flatness from the base manifold.

With respect to this, one can raise the following problem: does the moduli space M_k inherit the positivity (or negativity) of the first Chern class from the base manifold ?

For this problem we can say the following. On M_k a Kähler metric is defined naturally by means of the L_2 inner product over X. The curvature and hence the Ricci tensor of this metric are expressed in terms of integration over X by using the Green operator for an elliptic operator ([14]).

However, the Ricci form and hence $c_1(\widehat{M_k})$ can not in general be computed in a straightforward way.

In this paper we will discuss the positivity (resp., negativity) of the first Chern class $c_1(\widehat{M_k})$ by observing that it is the first Chern class of

Received March 10, 1989.