Advanced Studies in Pure Mathematics 18-I, 1990
Recent Topics in Differential and Analytic Geometry
pp. 251–270

Eta Invariants and Automorphisms of Compact Complex Manifolds

Akito Futaki and Kenji Tsuboi

Dedicated to Professor Akio Hattori on his sixtieth birthday

§1. Introduction

Let M be an m-dimensional compact complex manifold, G the group of all automorphisms of M and $\mathfrak{g}$ the complex Lie algebra of all holomorphic vector fields on M. In [F1, FMo] we defined a complex Lie algebra character $\mathcal{F} : \mathfrak{g} \to \mathbb{C}$ with properties that $\mathcal{F}$ depends only on the complex structure of M, and that the vanishing of $\mathcal{F}$ is a necessary condition for M to admit an Einstein-Kähler metric. $\mathcal{F}$ can be lifted to a group character $\widehat{\mathcal{F}} : G \to \mathbb{C}/\mathbb{Z}$. For these we refer the reader to a survey [FMaS], Chapters 1 and 3 in this volume; but brief reviews of $\mathcal{F}$ and $\widehat{\mathcal{F}}$ will be given respectively in this section and at the beginning of Section 3.

In this paper we apply the theory of eta invariants of [APS] and [D] to obtain an interpretation of $\widehat{\mathcal{F}}$ in terms of eta invariants (Theorem 3.7) and a localization formula for $\widehat{\mathcal{F}}(a)$ in terms of the fixed point set of an automorphism $a \in G$ (Theorem 3.10). We also compute a few examples.

An unsolved question, which motivated this study, is whether M admits an Einstein-Kähler metric if $c_1(M) > 0$ and $\mathfrak{g} = \{0\}$. Note that if $\mathfrak{g} = \{0\}$ then $\mathcal{F} = 0$ trivially. Our study began with an attempt to know whether $\widehat{\mathcal{F}}$ can play any role even if $\mathfrak{g} = \{0\}$. If $c_1(M) > 0$ and $\mathfrak{g} = \{0\}$ then G is a finite group and the imaginary part $\operatorname{Im} \widehat{\mathcal{F}} : G \to \mathbb{R}$ vanishes identically, but the real part $\operatorname{Re} \widehat{\mathcal{F}} : G \to \mathbb{R}/\mathbb{Z}$ may not do. Our aim is therefore to find an example of a compact complex manifold with $\mathfrak{g} = \{0\}$ and with $\widehat{\mathcal{F}} \neq 0$. We mention however that it is not known whether $\widehat{\mathcal{F}} \neq 0$ implies the nonexistence of an Einstein-Kähler metric,

Received November 30, 1988.
Revised March 20, 1989.