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Tate-Shafarevich Groups of Elliptic Curves with Complex Multiplication

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Dedicated to Professor Kenkichi Iwasawa on his 70th birthday

If E is an elliptic curve defined over an imaginary quadratic field K, with complex multiplication by K, and if $L(E_{/K}, 1) \neq 0$, then the Tate-Shafarevich group $\text{Ш}(E_{/K})$ is finite. The proof of this statement in [8] is complicated by the necessity of studying the $\mathfrak{p}$-part of $\text{Ш}(E_{/K})$ for *all* primes $\mathfrak{p}$ of K. In fact the above theorem grew out of an earlier weaker result which, because it ignores a finite set of "bad" primes of K, is proved much more simply.

The purpose of the present paper is to give the original proof of this simpler result, Theorem 1 below. The proof contains the important ideas of the proof of Theorem A of [8], but is much clearer because many of the technical difficulties of [8] do not arise. Later in this section we will use Theorem 1 to obtain three examples of finite Tate-Shafarevich groups. This paper should be viewed as the predecessor of [8], and one would be well-advised to read this paper first.

Suppose E is an elliptic curve defined over an imaginary quadratic field $K \subset \boldsymbol{C}$, with complex multiplication by the ring of integers $\mathcal{O}$ of K. Fix an $\mathcal{O}$-generator $\Omega \in \boldsymbol{C}^\times$ of the period lattice of a minimal model of E, let ψ denote the Hecke character of K attached to E, $L(\psi, s)$ the corresponding Hecke L-function, and $L(E_{/K}, s)$ the L-function of E over K. Then $L(E_{/K}, s) = L(\psi, s)L(\bar{\psi}, s)$, $L(\bar{\psi}, 1)/\Omega \in K$, and $L(E_{/K}, 1) = 0 \Leftrightarrow L(\psi, 1) = 0 \Leftrightarrow L(\bar{\psi}, 1) = 0$.

Theorem 1. *Let E be an elliptic curve defined over an imaginary quadratic field K, with complex multiplication by K. Let $\mathfrak{p}$ be a prime of K where E has good reduction, and which does not divide $\#(\mathcal{O}^\times)$. If $\#(E(K)_{\text{torsion}})L(\bar{\psi}, 1)/\Omega \not\equiv 0 \pmod{\mathfrak{p}}$, then the $\mathfrak{p}$-part of $\text{Ш}(E_{/K})$ is zero. In particular if $L(\bar{\psi}, 1) \neq 0$ then the $\mathfrak{p}$-part of $\text{Ш}(E_{/K})$ is zero for all but finitely*

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