Advanced Studies in Pure Mathematics 17, 1989
Algebraic Number Theory — in honor of K. Iwasawa
pp. 309–330

Fine Estimates for the Growth of e_n in Z_p^d-Extensions

P. Monsky

For Professor Kenkichi Iwasawa on his seventieth birthday

Introduction

Fix a prime p and a positive integer d. Suppose that $k=k_0 \subset k_1 \subset \cdots$ is a tower of number fields and that k_n is Galois over k with Galois group a product of d copies of Z/p^n. Let L be the union of the k_n; it is a "Z_p^d-extension of k". We denote the exponent to which p appears in the class number of k_n by $e_n(L/k)$, or more briefly by e_n.

When $d=1$ we have Iwasawa's celebrated formula: $e_n = \mu p^n + \lambda n + \nu$, $n \gg 0$, where μ and λ are integers attached to the Iwasawa module of L/k. Suppose that $d>1$. Then in place of the Iwasawa module we have the Greenberg-Iwasawa module, X, a finitely generated torsion module over $\Lambda = Z_p[\![X_1, \cdots, X_d]\!]$, and progress towards proving analogues of Iwasawa's formula has been made by studying such modules. (See for example [1], [2], [3], [4]). The analysis started with [2], in which Cuoco treated the case $d=2$, attaching integers m_0 and l_0 to X and showing that $e_n = m_0 p^{2n} + l_0 n p^n + O(p^n)$. This result was generalized to arbitrary d in [1]. In that paper integers $m_0 = m_0(F)$ and $l_0 = l_0(F)$ were attached to a non-zero $F \in \Lambda$ as follows. Let $\bar{\Lambda} = \Lambda/p\Lambda$ and $\bar{E}$ be the closed multiplicative subgroup of $\bar{\Lambda}$ generated by the $1+X_j$; it is a free rank d Z_p-module. Write $F = p^s G$ where the image, $\bar{G}$, of G in $\bar{\Lambda}$ is non-zero and write $\bar{G}$ as the product of irreducible G_i. Then $m_0(F) = s$ and $l_0(F)$ is the number of indices i for which (G_i) is the image of (X_1) under an automorphism of $\bar{\Lambda}$ transforming $\bar{E}$ to itself. The fundamental Theorem I of [1] states the following. Let $F \in \Lambda$ be the "characteristic power series" of the Greenberg-Iwasawa Λ-module, X, attached to L/k; set $m_0 = m_0(F)$ and $l_0 = l_0(F)$. Then $e_n(L/k) = (m_0 p^n + l_0 n + O(1)) p^{(d-1)n}$.

Our goal in this paper is to refine the above Theorem I by proving that $e_n = (m_0 p^n + l_0 n + \alpha^*) p^{(d-1)n} + O(n p^{(d-2)n})$ for some real α^*. There is no easy description of α^* and in particular we do not know if it is always rational. We shall show however that it is rational if either $d=2$ or if X

Received July 26, 1987.