Advanced Studies in Pure Mathematics 17, 1989
Algebraic Number Theory — in honor of K. Iwasawa
pp. 289–308

Arrangements of Hyperplanes, Higher Braid Groups and Higher Bruhat Orders

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Dedicated to Kenkichi Iwasawa on his seventieth birthday

Introduction

Let z_i be coordinate functions on C^n. Consider the arrangement of hyperplanes $D_{ij}: z_i - z_j = 0$ in C^n and let $U = C^n - \bigcup D_{ij}$ be its complement. The fundamental group of U is called the (colored) braid group. This group and the topology of U has been studied in many papers.

The family $\{D_{ij}\}$ is a special example of hyperplane arrangements which we call discriminantal ones. This article is devoted to the study of the topological and combinatorial properties of the discriminantal arrangements.

Among the vast literature on arrangements of hyperplanes we can mention Cartier's Bourbaki report [1] and an important paper [5]. Recently their study was stimulated by the theory of multidimensional hypergeometric functions (cf. [8–10]) and certain models of quantum and statistic physics (see [6], [7] and the bibliography therein).

In section 1 of this paper we recall some results on the hyperplane arrangements, define discriminantal arrangements (considered previously in [6], [7] and [10]), define higher braid groups and calculate their nilpotent completions.

In section 2 we introduce posets $B(n, k)$. Their definition is motivated by combinatorics of the discriminantal arrangements. The poset $B(n, 1)$ is essentially the symmetric group S_n with its weak Bruhat order. We prove some fundamental properties of $B(n, k)$ including the higher analogs of the Coxeter relations.

The results of section 2 were previously announced in [6], [7].

Actually, the construction of section 2 defines on S_n a canonical structure of $(n-1)$-category, whose "1-coskelton" is the category associated to the weak Bruhat order. This $(n-1)$-category is introduced in the section 3. Its structure is closely related to the combinatorial structure of the convex closure of a general orbit of S_n in $\boldsymbol{R}^n$.

Received October 16, 1987.