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Remarks on the Theorems of Takagi and Furtwängler

Tomio Kubota

Dedicated to Professor Kenkichi Iwasawa on his 70th birthday

The present article contains a short exposition of the fact that the fundamental inequality in Takagi's class field theory is almost a corollary of Furtwängler's theorems as far as unrestricted use of idele-theoretic terminology is allowed.

Let K/F be a cyclic extension of a finite degree n over an algebraic number field of a finite degree. Denote by I_L, P_L and C_L, the idele group, the principal idele group, and the idele class group of an algebraic number field L, respectively. Then, Takagi's fundamental inequality is written as $(C_F: N_{K/F}C_K) \geqq n$. (See e.g. [2], p. 154).

Let, on the other hand, F be an algebraic number field containing the group $\mu_{(l)}$ of the l-th roots of unity with a prime number l. Then, it was shown by Furtwängler [1] that the following two theorems hold:

Theorem I). *Product formula $\prod_{\mathfrak{p}} (\alpha, \beta/\mathfrak{p})_l = 1$ of the norm residue symbol $(\alpha, \beta/\mathfrak{p})_l$, where $\alpha, \beta \in F^\times = F - \{0\}$, and $\mathfrak{p}$ runs through all places of F.*

Theorem II). *Principal genus theorem $H^{-1}(C_K, \mathrm{Gal}(K/F)) = 1$ for an arbitrary Kummer extension K/F of degree l, which says that $N_{K/F}a = 1$ for $a \in C_K$ entails $a = b^{1-S}$ with some $b \in C_K$, where S is a generator of the Galois group $\mathrm{Gal}(K/F)$.*

Furtwängler's theorems are fully idele-theoretic. The second theorem was originally stated by him in the form that an element of F which is everywhere a local norm from K is a global norm from K.

We shall show that these two theorems of Furtwängler easily yields Takagi's fundamental inequality.

Proposition 1. *Let K/F be a cyclic extension of a finite degree over an algebraic number field F of a finite degree, and let K' be an intermediate field of K/F. Assume that $H^{-1}(C_K, \mathrm{Gal}(K/K')) = 1$ and $H^{-1}(C_{K'}, \mathrm{Gal}(K'/F)) = 1$. Then, $H^{-1}(C_K, \mathrm{Gal}(K/F)) = 1$.*

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