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Perversity and Exponential Sums

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Respectfully dedicated to Kenkichi Iwasawa on his seventieth birthday

Introduction

This paper is devoted to the study of the mean absolute value of exponential sums. The situation we have in mind is the following. Fix an integer $r \geq 1$, and a closed subscheme X of $\mathbf{A}^r_{\mathbf{Z}} = \mathrm{Spec}(\mathbf{Z}[x_1, \cdots, x_r])$, the r-dimensional affine space over $\mathbf{Z}$. Suppose that the complex variety $X_{\mathbf{C}}$ is reduced and irreducible, of dimension $n \geq 1$. For each prime number p, and each r-tuple $(a) = (a_1, \cdots, a_r)$ of elements of $\mathbf{F}_p$, we denote by $S(p; (a))$ the exponential sum

$$S(p; (a)) := \sum_{(x) \text{ in } X(\mathbf{F}_p)} \exp((2\pi i/p)(\Sigma_i a_i x_i)).$$

For each prime number p we denote by $M(p)$ the mean absolute value of the "normalized" sums $S(p; (a))/(\sqrt{p})^n$;

$$M(p) := p^{-r} \sum_{(a) \text{ in } (\mathbf{F}_p)^r} |S(p; (a))/(\sqrt{p})^n|.$$

A useful way to think about $M(p)$ is this. For fixed p, we can view $S(p; (a))/(\sqrt{p})^n$ as a complex-valued function $f_p((a))$ on the finite abelian group $(\mathbf{F}_p)^r$ of (a)'s. If we endow this group with its normalized Haar measure of total mass one, then $M(p)$ is precisely the L^1 norm of the function f_p. The function f_p is, by its very definition, the Fourier transform of a certain function g_p on the Pontryagin dual group $\mathbf{A}^r(\mathbf{F}_p)$, namely the function $g_p := (\sqrt{p})^{-n} \times$ (the characteristic function of $X(\mathbf{F}_p)$).

The number of points in $X(\mathbf{F}_p)$ is p^n, up to an error term which is $O((\sqrt{p})^{2n-1})$. So with respect to the dual Haar measure on $\mathbf{A}^r(\mathbf{F}_p)$, which gives every point mass one, the L^2 norm of the function g_p is equal to $1 + O(1/\sqrt{p})$. So by Parseval, the L^2 norm of f_p is also equal to $1 + O(1/\sqrt{p})$. Since the L^1 norm is bounded by the L^2 norm for total mass one, it follows that $M(p)$ is bounded by $1 + O(1/\sqrt{p})$.

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