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On Functional Equations of Zeta Distributions

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Dedicated to Prof. Ichiro Satake on his sixtieth birthday

Introduction

Recent development in the theory of prehomogeneous vector spaces (in particular the works of Gyoja-Kawanaka [10] on prehomogeneous vector spaces defined over finite fields and of Igusa [17] on prehomogeneous vector spaces defined over $\mathfrak{p}$-adic number fields) has revealed a striking resemblance between the theories over finite fields, $\mathfrak{p}$-adic number fields, real and complex number fields and algebraic number fields, as is common in the theory of representations of algebraic groups.

Now we give a brief sketch of the fundamental theorem in the theory of prehomogeneous vector spaces. Let K be one of the fields mentioned above and $(\boldsymbol{G}, \rho, \boldsymbol{V})$ be a K-regular prehomogeneous vector space (satisfying some additional conditions, if necessary). Take K-irreducible polynomials $P_1, \cdots, P_n$ defining the K-irreducible hypersurfaces contained in the singular set $\boldsymbol{S}$. Let $\Omega(K^\times)$ be the set of quasi-characters of the multiplicative group $K^\times$ and $\mathscr{S}(V(K))$ the space of Schwartz-Bruhat functions on $V(K)$. For an $\omega \in \Omega(K^\times)^n$ we can define a tempered distribution (zeta distribution) $Z(\omega)$ on $V(K)$ by analytic continuation of the integral

$$Z(\omega)(\phi)=\int_{V(K)-S(K)} \prod_{i=1}^{n} \omega_i(P_i(x))\phi(x)d_V^\times(x) \qquad (\phi \in \mathscr{S}(V(K))),$$

where $d_V^\times(x)$ is a certain relatively $\boldsymbol{G}(K)$-invariant measure on $\boldsymbol{V}(K)-\boldsymbol{S}(K)$. Starting from the prehomogeneous vector space $(\boldsymbol{G}, \rho^*, \boldsymbol{V}^*)$ dual to $(\boldsymbol{G}, \rho, \boldsymbol{V})$, we can obtain a tempered distribution $Z^*(\omega)$ ($\omega \in \Omega(K^\times)^n$) on $V^*(K)$.

Roughly speaking, the fundamental theorem states that the Fourier transform of the tempered distribution $Z(\omega)$ coincides with $Z^*(\omega^*)$ for certain ω^* up to a constant multiple $\gamma(\omega)$ depending meromorphically on ω: $\hat{Z}(\omega)=\gamma(\omega)Z^*(\omega^*)$.

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