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# The Structure of the Icosahedral Modular Group

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***Dedicated to Prof. Friedrich Hirzebruch and Prof. Ichiro Satake on their sixtieth birthdays***

## Introduction

By the group in the title we mean the Hilbert modular group PSL(2, $\mathcal{O}$) where $\mathcal{O}$ is the ring of integers in the number field $Q(\sqrt{5})$. This naming here comes from the following fact: Hirzebruch [H] studied the irreducible action over $H\times H$ ($H:=\{z\in C;\ \mathrm{Im}(z)>0\}$) of the principal congruence subgroup $\Gamma=\Gamma(2)$ of $SL(2,\mathcal{O})$ associated with the prime ideal $(2)\subseteq\mathcal{O}$ and showed that thecom pactified quotient $\overline{H\times H/\Gamma}$ is equivariantly birational to the projective plane $P_2(C)$ acted nontrivially by the icosahedral group $\mathfrak{A}_5\cong PSL(2,\mathcal{O}/(2))$ ($\mathcal{O}/(2)\cong F_4$; $\mathfrak{A}_5$: the alternating group of degree five). $\Gamma$ can be regarded as a subgroup of $PSL(2,\mathcal{O})$ since $-1\notin\Gamma$. It acts even freely on $H\times H$ and the description in [H] is so explicit that one can describe the arithmetic group $\Gamma$ as some quotient group of $\pi_1$ of the complement of the icosahedral arrangement. This arrangement consists of the reflecting lines of fifteen involutions in $\mathfrak{A}_5$ and is therefore mapped into itself by the group. Thus, if one calculates $\pi_1$ of the quotient by $\mathfrak{A}_5$ of the complement, then one succeeds in determining the structure of $PSL(2,\mathcal{O})$. Since we have only a few Hilbert modular groups for which the group-structure is combinatorially described, this is naturally of interest and is the purpose of this note. The main result is stated as follows:

**Theorem.** *The icosahedral modular group $PSL(2,\mathcal{O})$ is isomorphic to the group generated by three elements $\alpha$, $\beta$, $\gamma$ with the fundamental relation;*

$$\begin{cases}\alpha\beta=\beta\alpha \quad \alpha\gamma\alpha=\gamma\alpha\gamma \quad \beta\gamma\beta\gamma\beta=\gamma\beta\gamma\beta\gamma\\ (\beta\gamma)^2=1 \quad (\alpha\gamma)^3=1.\end{cases}$$

*Moreover, if we define the homomorphism $h: PSL(2,\mathcal{O})\to\mathfrak{A}_5$ by setting* $h(\alpha)=(1,4)(2,5)$, $h(\beta)=(1,2)(4,5)$, $h(\gamma)=(1,4)(2,3)$, *then we obtain the exact sequence:*

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