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On Automorphism Groups of Positive Definite Binary Quaternion Hermitian Lattices and New Mass Formula

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Dedicated to Professor Ichiro Satake on his sixtieth birthday

In this paper, we shall give some general method how to calculate the multiplicity of a given finite group which appears as the automorphism groups of the lattices, up to isometry, in a fixed genus in a positive definite metric space, and apply it to the binary quaternion hermitian cases, motivated by the theory of supersingular abelian varieties developed in Katsura-Oort [12]. Our Main Theorems are Theorems 7.1 and 7.2 in § 7. More precisely, we shall consider the following problems. Let B be either the rational number field $\boldsymbol{Q}$, an imaginary quadratic field over $\boldsymbol{Q}$, or a definite quaternion algebra over $\boldsymbol{Q}$. Let (V, h) be a pair of a finite dimensional left B-vector space V over B and a positive definite hermitian metric h on B with respect to the unique positive involution of B. Denote by $G=G(V, h)$ the group of similitudes of (V, h);

$$G=\{g \in GL_B(V);\ h(xg, yg)=n(g)h(x, y)\ (x, y \in V) \text{ for some } n(g) \in \boldsymbol{Q}^{\times}\}.$$

Let $\mathscr{L}$ be a fixed genus of some lattices in V.

Problem 1. Calculate the class number $H=\sharp(\mathscr{L}/G)$ of $\mathscr{L}$.

It is known that Problem 1 can be solved at least in principle by means of the trace formula (cf. Hashimoto [3]), and some explicit calculations have been done by several mathematicians. Now, our main theme in this paper is the following Problem 2. Denote by $L_1, \cdots, L_H$ a complete set of representatives of the classes in $\mathscr{L}$. For each i $(1 \leqq i \leqq H)$, put

$$\Gamma_i=\mathrm{Aut}(L_i)=\{g \in G;\ L_i g=L_i\}.$$

It is easy to see that this is a finite group for each i, because of our assumption that h is positive definite.

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