Advanced Studies in Pure Mathematics 15, 1989
Automorphic Forms and Geometry of Arithmetic Varieties
pp. 41–64

On Dimension Formula for Siegel Modular Forms

Ryuji Tsushima

§ 0. Introduction

Let $\mathfrak{S}_g$ be the Siegel upper half plane of degree g, and let $Sp(g, \boldsymbol{Z})$ be the Siegel modular group of degree g. Let Γ be a subgroup of $Sp(g, \boldsymbol{Z})$ of finite index, and let μ be a holomorphic representation of $GL(g, \boldsymbol{C})$ into $GL(r, \boldsymbol{C})$. By an *automorphic form of type* μ with respect to Γ, we mean a holomorphic mapping f of $\mathfrak{S}_g$ to the r-dimensional complex vector space $\boldsymbol{C}^r$ which satisfies the following equalities:

$$f(M\langle Z\rangle)=\mu(CZ+D)f(Z),$$

for any $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Gamma$ and $Z \in \mathfrak{S}_g$, where $M\langle Z\rangle$ is defined to be $(AZ+B)(CZ+D)^{-1}$. (We need to assume the holomorphy of f at "cusps" if $g=1$.) We denote by $A_\mu(\Gamma)$ the complex vector space of automorphic forms of type μ with respect to Γ. It is known that $A_\mu(\Gamma)$ is finite dimensional ([8]). In case $\mu(CZ+D)=\det(CZ+D)^k$, an automorphic form of type μ is also called an automorphic form of *weight* k, and $A_\mu(\Gamma)$ is also denoted by $A_k(\Gamma)$. In case the degree of μ is greater than one, an automorphic form of type μ is called a *vector-valued* automorphic form.

Our main problem is to find a formula for $\dim A_\mu(\Gamma)$ as a function in the signature of μ. The first result to our main problem was obtained by J.-I. Igusa in case $g=2$. (The case $g=1$ is classical.) Let $\Gamma_g(N)$ be the principal congruence subgroup of $Sp(g, \boldsymbol{Z})$ of level N, i.e.,

$$\Gamma_g(N)=\{M \in Sp(g, \boldsymbol{Z}) \mid M\equiv \boldsymbol{1}_{2g} \bmod N\}.$$

$A(\Gamma):=\oplus_{k\geq 0} A_k(\Gamma)$ has a structure of a graded ring. By using the theory of theta series, he explicitly determined the generators of $A(\Gamma)$ and represented these generators by theta constants for some cases such as $\Gamma=\Gamma_2(1)$ or $\Gamma_2(2)$ ([29], [30]). Especially $\dim A_k(\Gamma_2(1))$ and $\dim A_k(\Gamma_2(2))$ were known in this work. In [31], he constructed a graded ring homomorphism ρ_g from a subring R_g of $A(\Gamma_g(1))$ to the graded ring $S(2, 2g+2)$ of binary

Received November 11, 1987.
Revised June 23, 1988.