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Boundedness of Certain Unitarizable Harish-Chandra Modules

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§ 0. Introduction

Let G be a connected real semisimple Lie group, Z be the center of G, K be a maximal compact subgroup of G modulo Z, $U(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_c$ of the Lie algebra $\mathfrak{g}$ of G and $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. An element X of $\mathfrak{g}$ defines vector fields $\pi(X)$ and $D_R(X)$ on G by

$$(\pi(X)\phi)(g)=\frac{d}{dt}\phi(e^{-tX}g)|_{t=0}$$

and

$$(D_R(X)\phi)(g)=\frac{d}{dt}\phi(ge^{tX})|_{t=0}$$

for $\phi \in C^\infty(G)$. Then π and D_R extend to algebra homomorphisms of $U(\mathfrak{g})$ to the algebra of differential operators on G. For an element x of G we also define an endomorphism $\pi(x)$ of $C^\infty(G)$ by $(\pi(x)\phi)(g)=\phi(x^{-1}g)$ for $\phi \in C^\infty(G)$.

Let f be an element of $C^\infty(G)$ or a column vector of elements of $C^\infty(G)$. Suppose f is left K-finite and $Z(\mathfrak{g})$-finite (i.e. $\dim \sum_{k\in K} C\pi(k)f<\infty$ and $\dim \pi(Z(\mathfrak{g}))f<\infty$). Put $V_f=\pi(U(\mathfrak{g}))f$. Then V_f is a $(\mathfrak{g}, K)$-module under π. Moreover we say that V_f is a unitarizable Harish-Chandra module if there exists a unitary representation (τ, E) of G with finite length (i.e. (τ, E) is isomorphic to a finite direct sum of irreducible unitary representations) such that V_f is isomorphic to the Harish-Chandra module of (τ, E). In this paper we consider the following problem:

Suppose V_f is a unitarizable Harish-Chandra module. Then is the function $f(g)$ bounded when g tends to a certain infinite point?

Of course if we do not impose any other assumption on f, we have nothing to conclude. We have in mind that f satisfies some more conditions, such as, f corresponds to a section of the G-homogeneous vector bundle associated to a representation of a certain subgroup of G and/or f

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