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Closure Relations for Orbits on Affine Symmetric Spaces under the Action of Minimal Parabolic Subgroups

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§ 1. Introduction

Let G be a connected Lie group, σ an involutive automorphism of G and H a subgroup of G such that $G_0^\sigma \subset H \subset G^\sigma$ where $G^\sigma = \{x \in G \mid \sigma x = x\}$ and G_0^σ is the connected component of G^σ containing the identity. Then the factor space $H\backslash G$ is called an affine symmetric space. We assume that G is real semisimple throughout this paper.

Let P^0 be a minimal parabolic subgroup of G. Then a parametrization of the double coset decomposition $H\backslash G/P^0$ is given in [1] and [2]. In this paper we study the closure relations for the double coset decomposition.

The result of this paper can be stated as follows. Let $\mathfrak{g}$ be the Lie algebra of G and σ the automorphism of $\mathfrak{g}$ induced from the automorphism σ of G. Let θ be a Cartan involution of $\mathfrak{g}$ such that $\sigma\theta = \theta\sigma$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ (resp. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$) be the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces for σ (resp. θ).

Let x be an arbitrary element of G. By Theorem 1 in [1], there exists an $h \in G_0^\sigma$ such that $P = hxP^0x^{-1}h^{-1}$ can be written as

$$P = P(\mathfrak{a}, \Sigma^+) = Z_G(\mathfrak{a}) \exp \mathfrak{n}$$

where $\mathfrak{a}$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$, Σ^+ is a positive system of the root system Σ of the pair $(\mathfrak{g}, \mathfrak{a})$, $Z_G(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in G and $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}(\mathfrak{a}; \alpha)$. ($\mathfrak{g}(\mathfrak{a}; \alpha) = \{X \in \mathfrak{g} \mid [Y, X] = \alpha(Y)X$ for all $Y \in \mathfrak{a}\}$.) Since $(HxP^0)^{cl} = (HP)^{cl}hx$, we have only to study $(HP)^{cl}$.

Let K be the analytic subgroup of G for $\mathfrak{k}$ and put $H^a = (K \cap H) \cdot \exp(\mathfrak{p} \cap \mathfrak{q})$. Then $H^a\backslash G$ is called the affine symmetric space associated to $H\backslash G$ ([1]). For a subset S of G, we put $S^{op} = \{y \in G \mid (H^a y P)^{cl} \cap S \neq \emptyset\}$. Then it is clear that S^{op} is the minimal H^a-P invariant open subset of G containing S since the number of H^a-P double cosets in G is finite. For each root α in Σ, put $\mathfrak{a}^\alpha = \{Y \in \mathfrak{a} \mid \alpha(Y) = 0\}$, put $L_\alpha = Z_G(\mathfrak{a}^\alpha)$ and choose an element w_α of $N_K(\mathfrak{a})$ such that $\operatorname{Ad}(w_\alpha)|_{\mathfrak{a}}$ is the reflection with respect to α.

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