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A Survey of the Generalized Geroch Conjecture

H. Doi and K. Okamoto

Dedicated to Professor M. Sugiura for his 60th birthday

Introduction

It is widely admitted that to reach the true world of unitary representations of semisimple Lie groups we had to have some impact from the outside of mathematics. That was "Quantum Mechanics". The quantum mechanics has now developed into the "Quantum Field Theory". During recent years there has been considerable activity in the study of the group theoretical approach to the quantum field theory which suggests a new "Theory of group representations". Unfortunately, however, we do not have enough mathematical structures to get an insight into the new world.

On the other hand the quantum field theory has another origin, "The Classical Field Theory". The group theoretical aspect of the recent study of field equations shows that one should consider not merely finite but also infinite dimensional Lie groups and their homogeneous spaces. To mention only two, the theorem of Sato-Sato [17] says that the space of all the local solutions of the KP equations and their hierarchy is parametrized by the closure of the infinite dimensional Grassmann manifold, and the Geroch conjecture which was proved affirmatively by I. Hauser and F. J. Ernst [8] says that the Geroch group acts transitively, up to gauge transformations, all the local solutions of the stationary axisymmetric Einstein field equations. I. Hauser and F. J. Ernst have extended their work to the case of N Abelian gauge fields interacting with the gravitational field in an astonishingly beautiful way that it contains the vacuum case ($N=0$) and the Einstein-Maxwell field equations ($N=1$) [10].

The aim of this paper is to give a more or less self-contained exposition of the mathematically beautiful work of Hauser-Ernst on the generalized Geroch conjecture [10] which, we believe, attracts many mathematicians and is promising of further developments as a new branch of mathematics.

Roughly speaking the generalized Geroch conjecture asserts, that the $SU(N+1, 1)$ "Kac-Moody Lie group" acts transitively on the "moduli

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