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Geometric Constructions of Representations

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§ 1. Introduction

Beginning with the work of Gelfand, it has become apparent that there is a close connection between representations of a Lie group G and its coadjoint orbits, i.e., G-orbits in the dual of the Lie algebra. In the case of a nilpotent group, unitary representations correspond to coadjoint orbits equipped with real polarizations, and the correspondence was used by Kirillov [11] to actually construct the representations. Harish-Chandra's parametrization of those unitary representations which enter the Plancherel decomposition of $L^2(G)$, with G semisimple, can also be phrased in terms of coadjoint orbits, though his construction ties the representations only indirectly to the orbits in question. A direct geometric construction via coadjoint orbits was conjectured by Langlands [14] and carried out in [17–20, 25] — at least for the discrete series, but implicitly for the various other non-degenerate series as well. In this connection I should mention also Duflo's synthesis of the nilpotent and semisimple cases [5], which attaches unitary representations to coadjoint orbits for algebraic groups over $\boldsymbol{R}$.

A short note of Kostant [13] suggests a method for associating representations — not necessarily unitary representation — to G-orbits in the dual of the complexified Lie algebra. Attempts to carry out his program in practice quickly lead to major analytic difficulties, especially if the orbits carry polarizations that are neither maximally real nor maximally complex (the terminology will be explained in Section 3 below). Perhaps for this reason, among others, coadjoint orbits with arbitrary polarizations have received little attention. Zuckerman's derived functor construction [23] mimics the "orbit method" (for semisimple coadjoint orbits of semisimple Lie groups) algebraically, and thus avoids all analytic difficulties. The derived functor construction, too, has been used almost exclusively in the setting of maximally real or maximally complex polarizations; indeed, these very special polarizations suffice to obtain all irreducible Harish-Chandra modules [15, 23]. Nonetheless a case can be made

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