Advanced Studies in Pure Mathematics 14, 1988
Representations of Lie Groups, Kyoto, Hiroshima, 1986
pp. 123–151

Lie Algebra Cohomology and Holomorphic Continuation of Generalized Jacquet Integrals

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Introduction

In this paper there are two types of theorems. The first are generalizations of vanishing theorems of Kostant [K] and Lynch [L]. The second are the holomorphic continuation of certain integrals. The proofs of the two seemingly unrelated types of results have in common the use of certain operators, Q_j which are non-commutative analogues of the standard Euler operator on the space of polynomials in several variables.

We now describe an important class of examples of the results. Let G be a real reductive group of inner type. Let $\mathfrak{g}$ denote the Lie algebra of G and let Y be a nilpotent element in $\mathfrak{g}$. Then ([Ja, p. 99, Lemma 8]) there exist elements $X, H \in \mathfrak{g}$ such that $[X, Y]=H$ and $[H, X]=2X$, $[H, Y]=-2Y$. Fix a Cartan involution such that $\theta H=-H$. Let $\mathfrak{u}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{u}_2=\{x \in \mathfrak{g} \mid [H, x]=2x\}$. Let $z \in \mathbf{C}-\{0\}$ and let $\psi(x)=zB(Y, x)$ for $x \in \mathfrak{u}$. If V is a $\mathfrak{g}$-module then we define a new action π_ψ of $\mathfrak{u}$ on V by $\pi_\psi(x)v=xv-\psi(x)v$. We denote this $\mathfrak{u}$-module by $V\otimes\mathbf{C}_\psi$. The main theorem on Lie algebra cohomology implies

Theorem. *Let V be a $\mathfrak{g}$-module such that if $v \in V$ then $\pi_\psi(x)^k v=0$ for all $x \in \mathfrak{u}$ for some $k=k(v)$. Then $H^i(\mathfrak{u}, V\otimes \mathbf{C}_\psi)=(0)$ for $i>0$.*

We now describe the other type of results. Let $\mathfrak{p}$ be the sum of the eigenspaces for $\operatorname{ad} H$ with non-negative eigenvalue. Let $P=\{g \in G \mid \operatorname{Ad}(g)\mathfrak{p}\subset\mathfrak{p}\}$ and put $M=P\cap\theta(P)$. Let (σ, H_σ) be a finite dimensional irreducible representation of M. Put $\mathfrak{a}=\{Z \in \mathfrak{m} \mid [Z, \mathfrak{m}]=0,\ \theta Z=-Z\}$. If $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ then let $(\pi_{P,\sigma,\nu}, I_{P,\sigma}^\infty)$ be the corresponding (degenerate) principal series representation (see § 6). Let $\tau=\exp(\pi/2(x-Y))$.

The analytic results involve the study of integrals of the form

$$(*)\qquad J_{P,\sigma,\nu}(f)=\int_{\mathfrak{u}} e^{iB(Y,u)}f_{\sigma,\nu}(\tau\exp(u))du.$$

Received December 26, 1986.
1) Research partially supported by a grant from the National Science Foundation.