Advanced Studies in Pure Mathematics 13, 1988
Investigations in Number Theory
pp. 433–460

Local Densities of Quadratic Forms

Yoshiyuki Kitaoka

Dedicated to Professor I. Satake on his 60th birthday

Introduction

Let $A^{(m)}$, $B^{(n)}$ be integral positive definite matrices. Our problem is to study when the quadratic equation $A[X]=B$ has an integral solution. We know already that $A[X]=B$ has an integral solution provided that $m\geqq 2n+3$, it has an integral solution over Z_p and $\min_{0\neq x\in Z^n} B[x]$ is sufficiently large. But we know nothing about this problem for $m\leqq 2n+2$ except in the case of $n=1$. To have a perspective, we know empirically that it is better to study the magnitude of the number $r(B, A)$ of integral solutions of $A[X]=B$. Siegel showed that the weighted average of $r(B, A_i)$ for $A_i \in \operatorname{gen} A$ is an infinite product of the amount $\alpha_p(B, A)$ of local solutions, roughly speaking. Hence the local density $\alpha_p(B, A)$ may suggest something global. If, for example, the average is relatively large, that is, $\prod_p \alpha_p(B, A)>\kappa(>0)$, then we can expect $r(B, A')>0$ for every A' in $\operatorname{gen} A$. If, to the contrary, the average is relatively small, then we may expect that it is almost equal to $r(B, A'')$ for some A'' in $\operatorname{gen} A$, in other words, $r(B, A')/r(B, A'')$ may be sufficiently small for every A' in $\operatorname{gen} A$ with $\operatorname{cls} A'\neq \operatorname{cls} A''$, and it leads us to the linear independence of theta series like in the case of $m=n+1$ (cf. the conjectures in [2, 3, 13]). Although there is a gap between the behaviour of the infinite product $\prod_p \alpha_p(B, A)$ and the one of each $\alpha_p(B, A)$, we want to give sufficient conditions in order that $\lim_i \alpha_p(B_i, A)=0$ or $\lim_i \inf \alpha_p(B_i, A)>0$ at the outset.

Theorem A. *Let M, $N=N_1\perp N_2$ be regular quadratc lattices over Z_p and let $\{M_i\}_{i=1}^s$ be representatives of submodules in M isometric to N_1 which are not transformed mutually by isometries of M. Then there are positive constants $c_i(N_1, M_i)$ such that*

$$\alpha_p(N, M)=\sum_i c_i(N_1, M_i)\alpha_p(N_2, M_i^\perp).$$

Hence the behaviour of $\alpha_p(N_1\perp N_2, M)$ with N_1 fixed is reduced to the one of $\alpha_p(N_2, M_i^\perp)$.

Received October 7, 1986.