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On the Eighth Power Residue of Totally Positive Quadratic Units

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§ 0. Introduction

Let p be a prime number which is congruent to 3 modulo 4 and ε_p the totally positive fundamental unit of the real quadratic field $F=\boldsymbol{Q}(\sqrt{p})$. Let q be a prime number which is split in F and is congruent to 1 modulo 2^n. Then we may define 2^n-th power residue symbol $(\varepsilon_p/q)_{2^n}$ of ε_p modulo q as follows. For a prime factor $\mathcal{Q}$ of q in F, we choose an integer A such that

$$\varepsilon_p \equiv A \qquad \mod \mathcal{Q}.$$

The integer A is uniquely determined modulo q. The symbol $(\varepsilon_p/q)_{2^n}$ is defined only when A is a 2^{n-1}-th power residue modulo q and given by

$$(\varepsilon_p/q)_{2^n}=\begin{cases} 1 & \text{if } A \text{ is a } 2^n\text{-th power residue modulo } q, \\ -1 & \text{otherwise.}\end{cases}$$

This definition is independent of the choice of the prime ideal $\mathcal{Q}$ and the assumption imposed on q implies the following equivalence:

$(\varepsilon_p/q)_{2^n}=1 \Longleftrightarrow$ the polynomial $x^{2^n}-A$ factors into a product of distinct 2^n linear polynomials modulo q.

The symbol $(\varepsilon_p/q)_2$ (resp. $(\varepsilon_p/q)_4$) is usually called the quadratic symbol (resp. biquadratic symbol or quartic symbol) of ε_p modulo q. For the given q, it is comparatively easy to determine the sign of the quadratic symbol. Thus we have

$$(\varepsilon_p/q)_2=1 \Longleftrightarrow q\equiv 1 \mod 8.$$

The evaluation of the quartic residue symbol $(\varepsilon_p/q)_4$ are studied by many authors ([1], [2], [3], [4], [5], [7]). Here we shall quote one of their results. Let r be any positive odd multiples of the class number of the imaginary

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