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On the Decomposition Laws of Rational Primes in Certain Class 2 Extensions

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Introduction

By '*class* 2 *extension*' we understand throughout this paper Galois extension whose group is a finite nilpotent group of class 2.

The investigation of non-Abelian laws of prime decomposition in certain class 2 extensions over the rational number field Q has been made by a number of writers. The first results connecting this subject matter were obtained by Rédei [36]. He defined a symbol $\{a_1, a_2, a_3\}$ with $a_i \equiv 1 \pmod 4$ which expresses the prime decomposition in a certain non-Abelian Galois extension containing $Q(\sqrt{a_1}, \sqrt{a_2})$ of degree 8 over Q, and found the multiplication and inversion properties of the symbol. Kuroda [32] proved a reciprocity of the biquadratic residue symbol, and first discovered the decomposition laws of rational primes in certain non-Abelian Galois extensions containing $Q(\sqrt{-1})$ of degree 8 over Q in terms of biquadratic residue symbols. Furuta [10] generalized the reciprocity of Kuroda to the case of 2^n-th power residue symbol. Fröhlich [7] gave a general theory of the restricted biquadratic residue symbol, and discussed again Kuroda's results with deeper properties of the symbol. Fröhlich [9] defined a new symbol $[a_1, a_2, a]_c$ which coincides with Rédei's one for a certain fixed value of c, where $c \in H^2(G(K/Q), \{\pm 1\})$ and $K = Q(\sqrt{a_1}, \sqrt{a_2})$, and which expresses the prime decomposition in a certain non-Abelian Galois extension $\hat{K}$ containing K of degree 8 over Q associated with c. This symbol is essentially the same as the Artin symbol $\left(\frac{\hat{K}/K}{\mathfrak{A}}\right)$, $\mathfrak{A}$ being an ideal of K whose norm to Q is equal to (a). Under the restriction of $a_1 \equiv a_2 \equiv 1 \pmod 4$, he proved the decomposition theorems, the uniqueness theorems, the inversion laws and the multiplication laws for the symbol, and furthermore, stated without proof the explicit form for the symbol in terms of rational quadratic characters associated with certain rational ternary quadratic forms. Recently Furuta [13] defined a simpler symbol $[a_1, a_2, a]$ via a sufficiently large ray class field of $Q(\sqrt{a_1}, \sqrt{a_2})$, which is the same as

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