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On Kronecker's Limit Formula for Certain Biquadratic Fields

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§ 1. Introduction

In [1], Asai studied the Kronecker's limit formula for the Eisenstein series associated to an algebraic number field of class number one. He obtained the function $h(\xi)$ as an analogy of $\log|\eta(z)|$ and he showed that $h(\xi)$ satisfies certain differential equation and transformation formula. Recently, Elstrodt, Grunewald and Mennicke [2], generalized the above limit formula for the case of imaginary quadratic fields with arbitrary class number. They also showed many interesting formulas for the function associated to $h(\xi)$.

Our aim in this paper is to consider the Kronecker's limit formula for the zeta-function of certain biquadratic fields in connection with $h(\xi)$. To be more precise, let L be the composite of two imaginary quadratic fields k and K. We assume that the class number of k is one and that the discriminants of k and K have no common factor. Let C be any absolute ideal class of L and let $\zeta_L(s, C)$ be the zeta-function of the class C. We show that the limit formula for $\zeta_L(s, C)$ can be written by means of the curvilinear integral of the function $h(\xi)$. Here the curve is a semi-circle in $\boldsymbol{R}^3$ cannonically associated to the ideal class C (§ 4, Theorem 1). Since the Fourier coefficients of $h(\xi)$ are given by modified Bessel function, each term of the integral decreases rapidly. In Section 5, we give an approximation formula for each term appearing in the limit formula (§ 5, Theorems 2, 3). Finally using the table for the modified Bessel function ([6]), we give the approximate values for the integrals in the case $L=\boldsymbol{Q}(\sqrt{-4}, \sqrt{-3})$.

Notations. We denote by $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$, respectively, the rational number field, the real number field, and the complex number field. For an associative ring A with an identity, $A^\times$ denote the group of invertible elements. For $z \in \boldsymbol{C}$, $z \to \bar{z}$ denotes the complex conjugation, $S(z)=z+\bar{z}$ and $|z|^2=z\bar{z}$. For an algebraic extension X of Y, $N_{X/Y}$ means the relative norm.

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