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Quadratic Units and Congruences between Hilbert Modular Forms

Noburo Ishii

Introduction

Let F be a real quadratic field which has the totally positive fundamental unit. We put $F=\mathbf{Q}(\sqrt{m})$ with a positive square free integer m. We denote by $[1, \sqrt{m}]$ the order of F generated by 1 and $\sqrt{m}$ over the ring of integers $\mathbf{Z}$. Let ε_m be the smallest unit of F such that $\varepsilon_m>1$ and $\varepsilon_m \in [1, \sqrt{m}]$. We denote by K the number field generated by $\sqrt{-1}$ and $\sqrt[4]{\varepsilon_m}$ over the rational number field $\mathbf{Q}$ and by E the elliptic curve over F defined by the Weierstrass equation;

$$y^2=x^3+4\varepsilon_m x.$$

We can attach to K (resp. to E) Hilbert modular forms over F of weight one (resp. of weight two) in a natural way.

The aim of the present paper is to show that the "quartic residuacity" of ε_m provides congruences between these Hilbert modular forms. Further we calculate their Fourier coefficients and express the decomposition law between K and F by them.

§ 1. Hilbert modular forms

Let the notation be as in introduction. Denote by G the galois group of the normal extension K of $\mathbf{Q}$. Then G is of order 16 and is generated by the following three isomorphisms σ, φ and ρ:

$$\sigma(\sqrt[4]{\varepsilon_m})=\sqrt{-1}\,\sqrt[4]{\varepsilon_m}, \qquad \sigma(\sqrt{-1})=\sqrt{-1};$$
$$\varphi(\sqrt[4]{\varepsilon_m})=1/\sqrt[4]{\varepsilon_m}, \qquad \varphi(\sqrt{-1})=\sqrt{-1};$$
$$\rho(\sqrt[4]{\varepsilon_m})=\sqrt[4]{\varepsilon_m}, \qquad \rho(\sqrt{-1})=-\sqrt{-1}.$$

It is easy to see that they satisfy the relation;

$$\sigma^4=\varphi^2=\rho^2=1, \quad \varphi\sigma\varphi=\rho\sigma\rho=\sigma^3, \quad \varphi\rho=\rho\varphi.$$

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