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Arithmetic of Some Zeta Function Connected with the Eigenvalues of the Laplace-Beltrami Operator

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§ 1. Introduction

Let $\lambda_0=0\leqq\lambda_1\leqq\lambda_2\leqq\cdots$ run over the eigenvalues of the discrete spectrum of the Laplace-Beltrami operator on $L^2(H/\Gamma)$, where H is the upper half of the complex plane and we take $\Gamma=PSL(2, Z)$. It is well known that $\lambda_1>\frac{1}{4}$. We put $\lambda_j=\frac{1}{4}+r_j^2$ for $j\geqq 0$. In our previous work [7], we have introduced and studied the zeta function defined by

$$Z_\alpha(s)=\sum_{r_j>0}\frac{\sin(\alpha r_j)}{r_j^s},$$

where α is any positive number and the series is convergent for $\mathrm{Re}\, s>1$. Using the Selberg's trace formula, which will be stated below, we have shown that $Z_\alpha(s)$ is an entire function for any positive α. In this paper we are concerned with the arithmetical properties of the values of $Z_\alpha(s)$ at $s=1$ or $s=0$. In particular, we obtain some new expressions of the values of Dirichlet L-functions at $s=1$ and a new proof of Dirichlet's class number formula for the real quadratic number fields.

To explain a general principle, we recall a primitive situation. Let $\zeta(s)$ be the Riemann zeta function and let γ run over the positive imaginary parts of the zeros of $\zeta(s)$. We have introduced in [5] the zeta function defined by

$$\zeta_\alpha(s)=\sum_{\gamma>0}\frac{\sin(\alpha\gamma)}{\gamma^s},$$

where α is any positive number and the series is convergent for $\mathrm{Re}\, s>0$. We have shown under the Riemann Hypothesis that this is entire for any positive α. The value of $\zeta_\alpha(s)$ at $s=1$ has been known long before by Guinand [9]. Namely, $\zeta_\alpha(1)$ as $\alpha\to\infty$, is essentially

$$-\tfrac{1}{2}e^{-(1/2)\alpha}\Big(\sum_{n\leqq e^\alpha}\Lambda(n)-e^\alpha\Big),$$

where $\Lambda(n)$ is the von Mangoldt function defined by

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