Advanced Studies in Pure Mathematics 13, 1988
Investigations in Number Theory
pp. 215–236

Some Problems of Diophantine Approximation and a Kronecker's Limit Formula

Akio Fujii

§ 1. Introduction

Let α be a positive irrational number. It has been the subject of many mathematicians (e.g. Sierpinski [24], Lerch [20], Weyl [30], Hecke [12], Hardy-Littlewood [8]–[11], Behnke [4] [5], Ostrowski [21], Spencer [27], Sós [25] [26], Kesten [15], Erdös [6], Lang [19] and $\cdots$) to study as precise as possible the asymptotic behavior of the sum

$$\sum_{n\leqq X}\left(\{\alpha n\}-\frac{1}{2}\right)$$

as X tends to ∞, where $\{y\}$ is the fractional part of y and n runs over the integers $\geqq 1$. It does not seem that even for a quadratic irrational α this sum is understood in a satisfactory way.

Towards this problem Hecke [12] has introduced and studied the zeta function defined by

$$Z_\alpha(s)=\sum_{n=1}^{\infty}\frac{\{\alpha n\}-\frac{1}{2}}{n^s} \qquad \text{for } \operatorname{Re}(s)>1.$$

If $\alpha=\sqrt{D}$ or $1/\sqrt{D}$, $D\equiv 2$ or $3 \pmod 4$ and D is a square free integer $\geqq 1$, then he has shown that $Z_\alpha(s)$ can be continued analytically to the whole complex plane with simple poles at most at the points

$$s=-2k\pm 2\pi i\frac{n}{\log \eta_D}, \qquad k, n=0, 1, 2, \cdots$$

where η_D is the fundamental unit of the quadratic number field $Q(\sqrt{D})$ or the square of it. As a result, he has obtained an explicit formula for the Riesz mean of the second order. Precisely, he has shown that for the above α and for any positive δ,

Received April 24, 1986.