Advanced Studies in Pure Mathematics 13, 1988
Investigations in Number Theory
pp. 25-170

On Diophantine Inequalities of Real Indefinite Quadratic Forms of Additive Type in Four Variables

Yoshinobu Nakai

Chapter 0. Introduction and Statements of the Result

0.1. We have a famous result of H. D. Kloosterman [17] on the solubility of the Diophatine equation

$$ax_1^2+bx_2^2+cx_3^2+dx_4^2=n.$$

There, the so-called "Kloosterman sum", with $e(\xi)=\exp(2\pi\sqrt{-1})$ for real ξ,

$$\sum_{x;\ 1\leqq x\leqq q,\ \ (x,q)=1} e\left(\frac{1}{q}(ax+b\bar{x})\right) \qquad (x\bar{x}\equiv 1 \bmod q)$$

was estimated non-trivially. The error term in his asymptotic expansion of the number of Diophantine solutions would have been of the same order as the expected main term, thereby giving no positive result, if the Kloosterman sums had been estimated trivially. On the other hand, we have a result of H. Davenport and H. Heilbronn [5], acertaining the non-trivial solubility of the Diophantine inequality

$$|\lambda_1x_1^2+\lambda_2x_2^2+\lambda_3x_3^2+\lambda_4x_4^2+\lambda_5x_5^2|<\varepsilon$$

of a real indefinite quadratic form, for an arbitrarily given positive small ε. The proof in [5] was based on an extention of the so-called "circle method" of G. H. Hardy and J. E. Littlewood and on a lemma (Cf. 4.3.9) on simultaneous Diophantine approximations with small denominators. Then, is it possible to treat the Diophantine inequality

$$(*) \qquad |\lambda_1x_1^2+\lambda_2x_2^2+\lambda_3x_3^2+\lambda_4x_4^2|<\varepsilon$$

Received May 9, 1985.