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Abundant Central Extensions and Genus Fields

Hiroshi Suzuki

Let k be a finite algebraic number field and K be a Galois extension of k of finite degree. Let L be a central extension of K/k, i.e. $Z(E)\supset A$, where $G=G(K/k)$, $A=G(L/K)$ and $E=G(L/k)$ are Galois groups and $Z(E)$ means the center of E.

The genus field in L is the subfield which corresponds to $A\cap E^c$, E^c being the commutator subgroup of E. Generally this group is a quotient group of the dual $M(G)=H^{-3}(G, \boldsymbol{Z})$ of the Schur multiplier $H^2(G, \boldsymbol{Q}/\boldsymbol{Z})$ of G, and the condition $A\cap E^c=M(G)$ is equivalent to the fact that L is abundant for K/k (i.e. $L\cdot k^{\mathrm{ab}}$ is the maximal central extension of K/k, where k^{ab} is the maximal abelian extension of k). We call the genus field trivial if the genus field coincides with K. In words of group theory, it means that E is a central-commutator extension of $M(G)$ by G.

Miyake [2] treats a problem when K/k has an abundant central extension with trivial genus field. And Miyake [3] gives sufficient conditions, which come from an equivalent condition of this problem, that is

$$\{\alpha\in N_{K/k}J_k \mid \alpha^n\in k^\times\}\subset N_{K/k}J_K\cap k^\times\cdot N_{K/k}\{\alpha\in J_K \mid \alpha^n\in K^\times I_G J_K\}$$

for all any factors n of $\exp M(G)$, where J_K is the idèle group of K, I_G is the augmentation ideal of $\boldsymbol{Z}[G]$ and $\exp M(G)$ is the exponent of $M(G)$.

Miyake considered in [4] the problem in local fields in detail, and he gives examples of K/k which has no abundant central extension with trivial genus field for lack of local one at a place where the decomposition group coincides with $G=G(K/k)$.

Even though G is given, in general, the Galois groups of abundant central extensions with the trivial genus fields are not determined uniquely. In this paper we shall consider conditions of the embedding-problem-type that gives the existence of a global abundant central extension with the trivial genus field whose Galois group is a given one, and give several examples of such a case. We give also an example of K/k which has no global abundant central extension with the trivial genus field but has local ones at all places of k.

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