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Swan Conductors with Differential Values

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In this paper, we give a refinement of the classical theory of Swan conductors for discrete valuation rings, and refer to geometric applications.

Classically, the Swan conductor of a character of the Galois group takes values in $\boldsymbol{Z}$. Our Swan conductor takes values in some extension S of $\boldsymbol{Z}$. Let K be a complete discrete valuation field. We consider the following two cases; totally ramified Galois extensions of K (called Case I in this paper), and Galois extensions of ramification index one whose residue extension is purely inseparable and generated by one element (called Case II). In Case I, our group S is $K^\times/U_K^{(1)}$ ($U_K^{(1)}$ is the group of units which are $\equiv 1 \bmod m_K$, the maximal ideal of K). In Case II, our S is a certain group isomorphic to $K^\times/U_K^{(1)} \oplus \boldsymbol{Z}$, and S has elements written as $[\omega]$ for some non-zero differentials ω of the residue field (this is the reason of the title of this paper). The principle is that for a Galois extension L/K and for $\sigma \in \mathrm{Gal}(L/K)$, $\sigma \neq 1$, it is fruitful to consider not only the ideal I_σ of O_L generated by $\{a-\sigma(a);\ a \in O_L\}$ as in the definition of the classical Swan character, but also the homomorphism

$$\varphi_\sigma\colon \Omega^1_{O_L/O_K} \longrightarrow I_\sigma/I_\sigma^2;\ a\,db \longmapsto a(b-\sigma(b)).$$

In this paper, we define our Swan character as the pair (I_σ, $\varphi_\sigma \bmod m_L$). (Perhaps, it will give a better theory to consider modulo higher powers of m_L).

Most classical results (relations with subgroups and quotient groups, the integrality of Hasse-Arf$\cdots$) are generalized to our Swan conductors. As in the classical case, our conductor is related to the local class field theory. If the residue field is finite, our Swan conductor of a wildly ramified character $\chi\colon \mathrm{Gal}(L/K) \to \boldsymbol{C}^\times$ of degree one describes not only the maximal integer i such that $\chi(U_K^{(i)}) \neq \{1\}$, but also the homomorphism $U_K^{(i)}/U_K^{(i+1)} \to \boldsymbol{C}^\times$ induced by χ. This relation is generalized to higher local fields (Theorem (3.7)) and essentially to all K (Theorem (3.6)).

As the Swan conductor, the different also has a refinement with value in S (§2). This "refined different" already appeared in the work of

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