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Vanishing Cycles and Differentials of Curves over a Discrete Valuation Ring

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In this paper, we study curves over a discrete valuation ring as a sequel to [13]. We study the relation of l-adic vanishing cycles and differentials, both of which represent how far curves are from being smooth. We compare the length of the cohomology of the torsion parts of the sheaves of differentials and the dimension or the "total dimension" of the cohomology of the sheaves of vanishing cycles. For this, we use a special differential on the special fiber called "the relative canonical differential" defined in Section 2. It gives the dimension of the space of vanishing cycles in a special case.

We always use the following notations and terminology. S: the spectrum of a strict local discrete valuation ring A with algebraically closed residue field k of $\mathrm{ch}=p\geqq 0$. s (resp. η): the closed (resp. the generic) point of S. S-curve: flat and separated S-scheme of finite type purely of relative dimension 1 such that the generic fiber is smooth over η.

A relation between vanishing cycles and differentials will be given by the following conjecture. Let $\Lambda := \mathbf{Q}_l$, where l is a prime number different from p and $R\phi\Lambda$ (resp. $R\psi\Lambda$) be the complex of the sheaves of the vanishing cycles (resp. the nearby cycles) (for the definition, see [4]). Let dimtot denote the total dimension i.e. $\dim_\Lambda + \mathrm{Sw}$, where Sw is the Swan conductor.

Conjecture (0.1). *Suppose X is a regular flat separated S-scheme of finite type and Z is a subscheme of X_s such that Z is proper over s and that $X-Z$ is smooth over S. Then,*

$$\operatorname{dimtot} R\Gamma(Z, R\phi\Lambda) = -\operatorname{length}_{\mathcal{O}_S} R\Gamma(Z, \Omega^{\cdot}_{X/S,\mathrm{tors}}).$$

Or equivalently,

$$\operatorname{dimtot} R\Gamma(Z, R\psi\Lambda) = \dim R\Gamma(Z, \Lambda) - \operatorname{length}_{\mathcal{O}_S} R\Gamma(Z, \Omega^{\cdot}_{X/S,\mathrm{tors}}).$$

Here the supports of $\Omega^{\cdot}_{X/S,\mathrm{tors}}$ and $R\phi\Lambda$ are included in Z. This conjecture generalizes that of P. Deligne, Conjecture 1.9 of [5], which treats

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