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On the Quotients of the Fundamental Group of an Algebraic Curve

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To the memory of Professor Takehiko Miyata

§1. Introduction

Let k be an algebraically closed field and X an irreducible complete non-singular algebraic curve over k. We denote by $\pi_1(X)$ the algebraic fundamental group of X (see [3, Exp. V]). The group $\pi_1(X)$ may be canonically identified with the Galois group $\mathrm{Gal}(k(X)^{\mathrm{ur}}/k(X))$, where $k(X)$ is the function field of X over k and $k(X)^{\mathrm{ur}}$ is the maximum unramified extension of $k(X)$. When char $k=0$, it is a classical fact that the structure of $\pi_1(X)$ is determined by the genus g of X. Namely $\pi_1(X)$ is isomorphic to $\hat{\Gamma}_g$, the pro-finite completion of the fundamental group Γ_g of a Riemann surface of genus g;

$$\Gamma_g=\langle a_1,\cdots, a_g, b_1,\cdots, b_g \mid a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}=1\rangle .$$

However when char $k>0$, the group $\pi_1(X)$ has not been determined yet. In particular, we do not know the set of all finite quotient groups of $\pi_1(X)$. (We know that there exists a surjective homomorphism $\hat{\Gamma}_g\to\pi_1(X)$ (see Grothendieck [3, Exp. X]), but to determine its kernel is a difficult open problem.)

In the previous paper [4], the author considered a finite étale Galois covering $Y\to X$ and determined the action of $G=\mathrm{Gal}(Y/X)$ on the space of holomorphic differentials on Y. As its consequence the following Theorem A was obtained ([4, Theorem 5]). Here the integer $t(G)$ is defined as the minimum number of generators of the $k[G]$-module $I_G=\{\sum_{\sigma\in G} a_\sigma\cdot\sigma \mid \sum_{\sigma\in G} a_\sigma=0\}$, the augmentation ideal of the group algebra $k[G]$.

Theorem A. *If a finite group G is a quotient of the pro-finite group $\pi_1(X)$, then we have $t(G)\leqq g$ (g is the genus of X).*

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