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Torsion Points on Curves

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§ 1.

Let C be a smooth complete curve defined over a field K. Let $\bar{K}$ denote the algebraic closure of K. We define an equivalence relation on $C(\bar{K})$ as follows. If $P, Q \in C(\bar{K})$, then we write $P \sim Q$ iff a positive integral multiple of the divisor $P-Q$ is principal. We call an equivalence class under this relation a *torsion packet*.

Suppose J is the Jacobian of C, $P \in C(\bar{K})$ and $i: (C, P) \to (J, 0)$ is an Albanese mapping. Then Abel's theorem implies $i^{-1}((i(C) \cap J_{\mathrm{Tor}})(\bar{K}))$ is the torsion packet containing P.

Examples. (i) $C = \boldsymbol{P}^1_K$ then $C(\bar{K})$ is the unique torsion packet on C.

(ii) C is an elliptic curve. Then the torsion packets are the sets $\{P+T: T \in C(\bar{K})_{\mathrm{Tor}}\}$ for $P \in C(\bar{K})$. Hence every torsion packet is infinite and if $\operatorname{char}(K)=0$ or K has positive transcendence degree, the number of non-trivial torsion packets is infinite.

(iii) K is a field of positive characteristic and transcendence degree 0. Then $C(\bar{K})$ is a torsion packet.

(iv) char $K=0$ and $g(C) \geq 2$, then Raynaud has proven that every torsion packet is finite [R-1] and if $g(C) \geq 3$ there are only finitely many non-trivial torsion packets [R-2].

(v) If $g(C)=2$ the morphism

$$\begin{aligned} C \times C &\longrightarrow J \\ (P, Q) &\longmapsto (P-Q) \end{aligned}$$

is surjective and since $\sharp J(\bar{K})_{\mathrm{Tor}} = \infty$, $\sharp\{(P, Q): P \neq Q, P \sim Q\} = \infty$. This, together with the previous example, implies that if $\operatorname{char}(K)=0$ the number of non-trivial torsion packets on C is infinite.

(vi) Suppose $K = \boldsymbol{Q}$, m is a positive integer and F_m is the complete projective curve with homogeneous equation

$$X^m + Y^m + Z^m = 0.$$

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