Advanced Studies in Pure Mathematics 12, 1987
Galois Representations and Arithmetic Algebraic Geometry
pp. 201–219

The Lower Central Series of the Pure Braid Group of an Algebraic Curve

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Dedicated to Professor Nagayoshi Iwahori on his 60-th birthday

§ 1. Main results

This short note contains four theorems: two results on the lower central series of the pure braid group of a Riemann surface and two arithmetic analogies of these theorems.

Let us describe the first two results. Let R be a Riemann surface of genus g, let $\prod_{i=1}^{n} R$ denote the n-fold product space of R, and let $F_{0,n}R$ denote the subspace

$$F_{0,n}R=\{(z_1, z_2, \cdots, z_n) \in \prod_{i=1}^{n} R \mid z_i \neq z_j, \text{ if } i \neq j\}.$$

The space $F_{0,n}R$ is a $K(\pi, 1)$-space, and the fundamental group of $F_{0,n}R$ is the pure braid group with n strings of the Riemann surface R (cf. Birman [2], Chap. 2).

In general, for a simplicial complex X we define the *holonomy Lie algebra* of X over $\boldsymbol{Q}$ in the following way (see [12]). Let

$$\eta\colon H_2(X; \boldsymbol{Q}) \longrightarrow \wedge^2 H_1(X; \boldsymbol{Q})$$

be the dual of the cup product homomorphism. Let $\mathscr{L}(H_1(X; \boldsymbol{Q}))$ be the free Lie algebra generated by $H_1(X; \boldsymbol{Q})$ over $\boldsymbol{Q}$. We identify the homogeneous part of degree 2 in $\mathscr{L}(H_1(X; \boldsymbol{Q}))$ with $\wedge^2 H_1(X; \boldsymbol{Q})$. We denote by J the homogeneous ideal of $\mathscr{L}(H_1(X; \boldsymbol{Q}))$ generated by the image of η. The holonomy Lie algebra $\mathfrak{g}_X$ over $\boldsymbol{Q}$ is defined to be $\mathscr{L}(H_1(X; \boldsymbol{Q}))/J$. Let

$$\Gamma_1\mathfrak{g}_X \supset \cdots \supset \Gamma_m\mathfrak{g}_X \supset \cdots$$

be the lower central series of $\mathfrak{g}_X$ defined recursively by

Received March 26, 1986.