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On the Automorphism Group of Some Pro-l Fundamental Groups

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Introduction

Let l be a fixed prime number and $g \geq 2$ be an integer. Let G be the pro-l completion of the fundamental group of a compact Riemann surface of genus g, i.e. G is a pro-l group generated by $2g$ elements $x_1, \cdots, x_{2g}$ with one defining relation;

$$(*) \qquad [x_1, x_{g+1}][x_2, x_{g+2}]\cdot \cdots \cdot [x_g, x_{2g}]=1$$
$$G=\langle x_1, x_2, \cdots, x_{2g} \,|\, [x_1, x_{g+1}]\cdot \cdots \cdot [x_g, x_{2g}]=1\rangle_{\text{pro-}l}.$$

([,] denotes the commutator; $[x, y]=xyx^{-1}y^{-1}$.) Let $\tilde{\Gamma}_g$ denote the group of continuous automorphism of G and Γ_g denote the outer automorphism group of G; $\Gamma_g=\tilde{\Gamma}_g/\mathrm{Int}\, G$, $\mathrm{Int}\, G$ being the inner automorphism group of G. (Note that every continuous automorphism of G is bi-continuous, as G is compact.) Our aim in this paper is to study these groups $\tilde{\Gamma}_g$ and Γ_g, as a generalization of Ihara [I_1] Chapter I and as a preliminary to the study of the Galois representations. We shall give filtrations of $\tilde{\Gamma}_g$ and Γ_g and prove a result on conjugacy classes of Γ_g.

Now we shall state our results. Let G_{ab} denote the abelianized group of G, so G_{ab} is a free $\boldsymbol{Z}_l$-module of rank $2g$ with a basis $\bar{x}_1, \cdots, \bar{x}_{2g}$. ($\boldsymbol{Z}_l$ denotes the ring of l-adic integers, and $\bar{x}_i$ denotes the class of x_i ($1 \leq i \leq 2g$).) The group $\tilde{\Gamma}_g$ acts on G_{ab} naturally and, with respect to the basis $\{\bar{x}_i\}_{1\leq i\leq 2g}$, we get a representation

$$\tilde{\lambda}: \tilde{\Gamma}_g \longrightarrow \mathrm{Aut}\, G_{\mathrm{ab}} \simeq \mathrm{GL}(2g; \boldsymbol{Z}_l).$$

The group $\tilde{\Gamma}_g$ also acts naturally on the cohomology group $H^i(G; \boldsymbol{Z}_l)$ ($i=1, 2$). (The action of G on $\boldsymbol{Z}_l$ is trivial.) Now the cup product

$$H^1(G; \boldsymbol{Z}_l)\times H^1(G; \boldsymbol{Z}_l) \longrightarrow H^2(G; \boldsymbol{Z}_l) \simeq \boldsymbol{Z}_l$$

defines a non-degenerate alternating form, and the action of $\tilde{\Gamma}_g$ on $H^i(G; \boldsymbol{Z}_l)$ ($i=1, 2$) are compatible with this cup product. It is well known

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