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Some Observations on Jacobi Sums

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Dedicated to Professor S. Iyanaga for his 80th birthday

In this paper, we shall be interested in the properties of the rational numbers $N(1-j(\alpha)q^{-n/2})$ which are the norm of the difference of 1 and $j(\alpha)q^{-n/2}$ (algebraic numbers of absolute value 1 in a cyclotomic field) where $j(\alpha)$ are the Jacobi sums defined below by (1.3), α being suitable $(n+2)$-tuples of integers modulo m. Typically, a formula of the form

$$N(1-j(\alpha)q^{-n/2})=(\text{square})(m\text{-power})/(q\text{-power})$$

seems to hold for m prime and n even, and we want to understand such a phenomenon in connection with arithmetic geometry of the Fermat varieties. The main result is Theorem 7.1 in Section 7 for the case $n=2$. The plan of the paper will be found at the end of Section 1.

Notation. Throughout the paper, the following notation will be used.
m: a fixed positive integer >2
$K=\boldsymbol{Q}(\zeta_m)$: the m-th cyclotomic field ($\zeta_m=e^{2\pi i/m}$)
n: a non-negative integer
$\mathfrak{A}_m^n$: the set of $(n+2)$-tuples $\alpha=(a_0, \cdots, a_{n+1})$ such that

$$a_i \in \boldsymbol{Z}/m, \quad a_i\neq 0, \quad \sum_{i=0}^{n+1} a_i=0$$

$\|\alpha\|=\sum_{i=0}^{n+1}\langle a_i/m\rangle-1$. ($\langle x\rangle$ is the fractional part of $x\in\boldsymbol{Q}/\boldsymbol{Z}$).
p: a prime number, $p\nmid m$.
$q=p^\nu$: a power of p such that $q\equiv 1 \pmod m$
$j(\alpha)$: the Jacobi sum (see § 1, (1.3))
$X_m^n(q)$: the Fermat variety $\sum_{i=0}^{n+1} x_i^m=0$ in $\boldsymbol{P}^{n+1}$ defined over $\boldsymbol{F}_q$.
$G_m^n=(\mu_m)^{n+1}/(\text{diagonal})$ regarded as a subgroup of $\mathrm{Aut}(X_m^n(q))$
$\hat{G}_m^n=$ the character group of G_m^n
$\quad=\{(a_0, \cdots, a_{n+1})\,|\, a_i\in\boldsymbol{Z}/m, \sum_{i=0}^{n+1} a_i=0\}$
$0(\hat{G}_m^n)$: the set of $(\boldsymbol{Z}/m)^\times$-orbits in $\hat{G}_m^n$
$0(\mathfrak{A}_m^n)$: 〃 $\mathfrak{A}_m^n$

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