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On the l-Adic Expansion of Certain Gauss Sums and Its Applications*)

Hiroo Miki

Introduction

In the present paper, we shall give a new explicit formula on the l-adic expansion (mod π^l) of certain Gauss sums (see Theorem 1 in Section 1).

Let p be any prime number and let $m>1$ be a natural number which is not divisible by p. Let ζ_m be a primitive m-th root of unity in the field of complex numbers $\boldsymbol{C}$. Let $\boldsymbol{Q}$ be the field of rational numbers and let $\boldsymbol{Z}$ be the ring of rational integers. Fix a prime ideal $\mathfrak{p}$ of $\boldsymbol{Q}(\zeta_m)$ lying above p and put $N\mathfrak{p}=q$, where $N\mathfrak{p}$ is the absolute norm of $\mathfrak{p}$. Note that $m|(q-1)$. Let $\boldsymbol{F}_q$ be the finite field of q elements. Let

$$\chi_{\mathfrak{p}}(x \bmod \mathfrak{p})=\left(\frac{x}{\mathfrak{p}}\right)_m$$

be the m-th power residue symbol in $\boldsymbol{Q}(\zeta_m)$, i.e.,

$$\chi_{\mathfrak{p}}(x \bmod \mathfrak{p})\equiv x^{(q-1)/m} \pmod{\mathfrak{p}}$$

for $x \in \boldsymbol{Z}[\zeta_m]$. $\chi_{\mathfrak{p}}$ induces a homomorphism of the multiplicative group $\boldsymbol{F}_q^\times$ of $\boldsymbol{F}_q$ to $\boldsymbol{C}^\times$ of order m and $\chi_{\mathfrak{p}}(0)=0$. Here we identify $\boldsymbol{F}_q$ and $\boldsymbol{Z}[\zeta_m]/\mathfrak{p}$. Let T be the trace of $\boldsymbol{F}_q$ to $\boldsymbol{F}_p$ and put

$$\psi(x)=\zeta_p^{T(x)}$$

for $x \in \boldsymbol{F}_q$. Then ψ is a homomorphism of the additive group $\boldsymbol{F}_q$ to the multiplicative group $\boldsymbol{C}^\times$.

Definition. For each $a \in \boldsymbol{Z}$, put

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