Advanced Studies in Pure Mathematics 12, 1987
Galois Representations and Arithmetic Algebraic Geometry
pp. 53–64

The Non-Vanishing of a Certain Kummer Character χ_m (after C. Soulé), and Some Related Topics

H. Ichimura and K. Sakaguchi

§ 1. Introduction

Let l be an odd prime number, and Ω_l be the maximum pro-l abelian extension of the cyclotomic field $\boldsymbol{Q}(\mu_{l^\infty})$ unramified outside l. This extension $\Omega_l/\boldsymbol{Q}(\mu_{l^\infty})$ and its Galois group $\mathfrak{G}$ are very important (and familiar) objects in Iwasawa theory, and the structure of $\mathfrak{G}$ as a $\boldsymbol{Z}_l^\times$ ($\cong \mathrm{Gal}(\boldsymbol{Q}(\mu_{l^\infty})/\boldsymbol{Q})$)-module has been investigated by Iwasawa ([Iw1] [Iw2]), Coates ([Coa]), Mazur and Wiles ([MW]), etc. In connection with Ihara's universal power series for Jacobi sums ([Ih] [IKY]), we are particularly interested in the groups

$$\mathrm{Hom}_{\boldsymbol{Z}_l^\times}(\mathfrak{G}, \boldsymbol{Z}_l(m)), \tag{1}$$

for positive integers m, and their "standard elements" χ_m which appear in Soulé [S3], Deligne [D] and Ihara [Ih]. The main purpose of this report is to give an exposition of the basic results, mainly due to Soulé [S2] [S3], on the structure of the group (1) and the non-vanishing of χ_m (for $m \geqq 3$, odd). The reason for writing this report is that, although these works seem to be well-known among the K-theorists, they are "buried" under a mass of generalities in K-theory and hence are not so familiar among the wider public in number theory. The second purpose of this report is to present various related remarks, in connection with Ihara's power series, the Vandiver conjecture, etc.

The first main theorem to be reviewed is the following.

Theorem A (Tate [T], Lichtenbaum [L], Soulé [S2]).

$$\mathrm{Hom}_{\boldsymbol{Z}_l^\times}(\mathfrak{G}, \boldsymbol{Z}_l(m)) \cong \begin{cases} \boldsymbol{Z}_l \cdots m \geqq 1, \textit{ odd} \\ \{0\} \cdots m \geqq 2, \textit{ even.} \end{cases}$$

To write down the second, let us recall the definition of the element $\chi_m \in \mathrm{Hom}_{\boldsymbol{Z}_l^\times}(\mathfrak{G}, \boldsymbol{Z}_l(m))$. Choose a basis $(\zeta_n)_{n\geq 1}$ of the Tate module

Received March 13, 1986.