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The Gross-Koblitz Formula

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§ 0. Introduction

Our intent in this note is to give a relatively self-contained and not completely ad hoc proof of the beautiful formula discovered by Gross and Koblitz [G-K] expressing Gauss sums in terms of Morita's p-adic gamma function. In fact we give two proofs.

The first is nothing more than a reorganization of that given by Boyarski in [B], who observed that the Gross-Koblitz formula could be recovered easily from results in a series of papers by Dwork. In this proof one makes use of Dwork cohomology spaces of the most elementary sort. These spaces are ultimately connected to the geometry of the Fermat curves but this connection is neither made explicit nor used in this note (see the appendix however). Unfortunately, this proof fails to work when $p=2$.

Our second proof was motivated by the first. It is more elementary in that it avoids the Dwork cohomology. Also, it does work when $p=2$.

A key ingredient in both proofs is formula (9) of Section II below, which is a formal version of Euler's integral representation of the classical gamma function. Other proofs of the Gross-Koblitz formula may be found in [K] and [L].

We will now state the Gross-Koblitz formula. Let $\boldsymbol{N}$ donote the natural numbers and $\boldsymbol{N}^*$ the positive integers. Let p be a prime. Let $\boldsymbol{C}_p$ denote the completion of an algebraic closure of $\boldsymbol{Q}_p$, the field of p-adic numbers. Let $q=p^f$ where $f \in \boldsymbol{N}^*$. Let $\boldsymbol{F}_q$ denote the field of order q contained in the residue field of $\boldsymbol{C}_p$. Let $t\colon \boldsymbol{F}_q \to \boldsymbol{C}_p$ denote the Teichmüller character on $\boldsymbol{F}_q$. That is, $t(x)$ is the unique element of $\boldsymbol{C}_p$ which reduces to x and satisfies $t(x)^q=t(x)$. It is a multiplicative character on $\boldsymbol{F}_q^*$. Fix a solution of $\pi^{p-1}=-p$ in $\boldsymbol{C}_p$, and let ζ_π denote the unique p^{th} root of unity in $\boldsymbol{C}_p$ satisfying

$$|\zeta_\pi-(1+\pi)|<|\pi|.$$

Set for $x \in \boldsymbol{F}_q$

$$\omega_q(x)=\zeta^{\operatorname{Trace}_{\boldsymbol{F}_q/\boldsymbol{F}_p}(x)}.$$

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