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Separated Ultraproducts and Big Cohen-Macaulay Modules

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This short note is meant to be a brief introduction to my theory which is developed in my paper, "Toward the construction of big Cohen-Macaulay modules." (Nagoya Math. J. 103 (1986), 96–125.) The reader should consult with my original paper for the details.

Section 1

Let $(T, \mathfrak{m})$ be an Artin local ring (commutative with unity) and let R be a T-algebra. We consider the following

Question. When does there exist a (non-trivial) T-injective R-module?

Roughly speaking my answer to this question is that *many T-algebras have such modules, while only a few do not.* The aim of this section is to explain this fact. For the convenience I make the following

Definition. A T-algebra R is a *rich* T-algebra if R has a T-injective R-module. Non-rich T-algebras are called *poor.*

The following lemma is an immediate consequence.

Lemma. *Flat* $\Longrightarrow$ *Rich* $\Longrightarrow$ *Pure.*

Proof. If R is T-flat and $E=E_T(T/\mathfrak{m})$, then $\mathrm{Hom}_T(R, E)$ is a T-injective R-module, hence R is a rich T-algebra. If M is a T-injective R-module, then M is a direct sum of copies of E as a T-module, so there is an injection of E into M. Taking the dual of this injection, one has the mapping f: $\mathrm{Hom}_T(M, E) \to \mathrm{Hom}_T(E, E)=T$. Thus there is an element x in $\mathrm{Hom}_T(M, E)$ satisfying $f(x)=1$. Define an R-module map g by $g(1)=x$. ($\mathrm{Hom}_T(M, E)$ is an R-module.) Then the composition $f\cdot g$ is a T-homomorphism of R onto T, hence T is a pure subring of R.

The converses of the implications in Lemma do not hold in general.

Example. (1) Let $T=k[\![x, w]\!]/(x^2, w^4)$ and let

$$R=k[\![x, y, z, w]\!]/(x^2, w^4, xw-yz, x^2z-y^3, yw^2-z^3, xz^2-y^2w)$$

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