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The Dilworth Number of Artinian Rings and Finite Posets with Rank Function

Junzo Watanabe

Introduction

In my paper [13] I proved that for any ideal $\mathfrak{a}$ of an Artinian local ring A and for any non-unit element y in A there is an inequality $\mu(\mathfrak{a}) \leqq l(A/yA)$. Thus we are naturally led to consider the numbers $d(A) := \mathrm{Max}\,\{\mu(\mathfrak{a})\}$ and $r(A) := \mathrm{Min}\,\{l(A/yA)\}$. The present paper has two purposes: (1) To give a combinatorial interpretation of the number $d(A)$ and (2) to study one case where the equality $d(A) = r(A)$ holds.

Although the problems concerning the number of generators of ideals have drawn considerable attention (for examples Sally [9]), the number $d(A)$ of an Artinian ring A does not seem to have ever been considered explicitly. But as soon as one tries to compute the number, taking an example of Artinian ring of "monomial type", one realizes that this is quite a combinatorial question, and fortunately some theorems and certain ideas in combinatorics are available for the purpose. To mention some of these, Dilworth's theorem, Sperner property and symmetric chain decomposition of posets. I called the number $d(A)$ the Dilworth number of the Artinian ring A because with an Artinian ring A of monomial type a poset is naturally associated and $d(A)$ coincides with what the combinatorists call the Dilworth number of the poset.

As to the number $r(A)$, I called it the Rees number because Rees [8] defined the notion of general elements of local rings in a general setting. The definition of a general element in the Artinian case adopted in [13] and in the present paper is slightly different from his: namely we say that y is a general element of A if $l(A/yA) = r(A)$ provided that A has an infinite residue field. The significance of this number is that it bounds the number of generators of ideals of the ring. I.e., $d(A) \leqq r(A)$. Now a natural question arises: when does the equality hold? To answer this question seems very difficult, and because a general theory cannot be expected at this time, what we do here is to consider a certain class of

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