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Linear Diophantine Equations and Invariant Theory of Matrices

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Introduction

In this paper, we shall study the Poincaré series of the ring of invariants of $n\times n$ matrices under the simultaneous adjoint action of $GL(n)$. This ring of invariants was studied by Procesi [3] and others. If $n=2$, it is well known that the ring of invariants of two generic matrices X and Y is a polynomial ring generated by 5 algebraically independent invariants

$$\mathrm{tr}(X),\ \mathrm{tr}(X^2),\ \mathrm{tr}(Y),\ \mathrm{tr}(Y^2),\ \mathrm{tr}(XY),$$

and hence the Poincaré series is

$$1/(1-s)(1-s^2)(1-t)(1-t^2)(1-st). \quad \text{(See [1]).}$$

However if $n\geq 3$, the ring of invariants is not polynomial ring. The Poincaré series of the ring of invariants for generic $n\times n$ matrices is related with the following generating function $F(t)$ of a linear diophantine equations defined by

$$F(t)=\sum_{r\geq 0} h(r)t^r,$$

where $h(r)$ is the number of $n\times n$ matrices $l=(l_{ij})\in M(n, N)$ with the property:

$$\sum_{i,j} l_{ij}=r \quad \text{and} \quad \sum_j l_{ij}=\sum_j l_{ji}, \quad 1\leq i\leq n.$$

General "reciprocity theorems" of the generating function of a linear diophantine equations is established by Stanley ([4], [5], [7]). We shall give simple proofs of some Stanley's results in [5].

By using a combinatorial method, we shall calculate the Poincaré series of the ring of invariants of two 4×4 generic matrices.

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