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Generalized H-Vectors, Intersection Cohomology of Toric Varieties, and Related Results

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§ 1. Background

Let $\mathscr{P}$ be a simplicial convex d-polytope, i.e., a d-dimensional convex polytope all of whose proper faces are simplices. Let $f_i = f_i(\mathscr{P})$ denote the number of i-dimensional faces of $\mathscr{P}$. The vector $f(\mathscr{P}) = (f_0, f_1, \cdots, f_{d-1})$ is called the *f-vector* of $\mathscr{P}$, and in [B-L] and [S₅] such vectors are completely characterized. A survey of this subject appears in [S₈]. Here we will be interested in extending the ideas of [S₈] to the non-simplicial case. While we come nowhere near a characterization of f-vectors for non-simplicial polytopes $\mathscr{P}$, we do discuss an interesting numerical sequence associated with $\mathscr{P}$. We also discuss some extensions of this work, as well as many conjectures and open problems. For instance, Conjectures 4.2 (b), 4.3 and 5.5 (a) extend the result that the h-vector of a Cohen-Macaulay complex is nonnegative.

Let us first review some material related to simplicial complexes, most of it to be found in [S₈]. Let Δ be an abstract $(d-1)$-dimensional simplicial complex on the n-element vertex set V, with $f_i = f_i(\Delta)$ i-dimensional faces (or faces with $(i+1)$-elements). Here the empty set $\emptyset$ is regarded as a face of dimension -1, so $f_{-1} = 1$. Define a vector $h(\Delta) = (h_0, h_1, \cdots, h_d)$, called the *h-vector* of Δ, by the condition

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}. \tag{1}$$

In particular,

$$h_0 = 1,\ h_1 = f_0 - d,\ h_d = (-1)^{d-1}\tilde{\chi}(\Delta),\quad \sum h_i = f_{d-1}, \tag{2}$$

where $\tilde{\chi}(\Delta)$ denotes the reduced Euler characteristic of Δ, i.e., $\tilde{\chi}(\Delta) = \sum_{i\geq -1}(-1)^i f_i$. The *Dehn-Sommerville equations* for the simplicial polytope $\mathscr{P}$ assert that $h_i = h_{d-i}$ when Δ is the boundary complex $\Delta(\mathscr{P})$ of $\mathscr{P}$ (so $f_i(\mathscr{P}) = f_i(\Delta)$, and we set $h_i(\mathscr{P}) = h_i(\Delta)$).

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