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Maximal Analytic Spread in Birational Extensions of Regular Local Rings

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The purpose of this note is to give a different viewpoint and a different proof for the result in [Sy] which states that if $(R, m)\subsetneq(S, n)$ are d-dimensional regular local rings with the same quotient field, then the analytic spread, $l(mS)$, is at most $d-1$. The different viewpoint comes from dropping the assumption that (S, n) is regular and seeing what it means for $l(mS)$ to be equal to d. This turns out to be a very useful tool for studying birational extensions of regular local rings. The different proof evolved after communications from Craig Huneke and David Rees. Huneke showed me that certain hypotheses in [Sy] implied that the natural map from R/m to S/n is an isomorphism.

Theorem. *Let (R, m) be a d-dimensional regular local ring. Let (S, n) be a d-dimensional local ring which birationally dominates R. If $l(mS)=d$, then*

(i) *S is dominated by the m-adic prime divisor of R*

(ii) *$R/m=S/n$*

(iii) *the natural map of the associated graded ring of R to the associated graded ring of S is injective.*

Note that (ii) and (iii) imply that a minimal basis for m in R is a subset of a minimal basis of any ideal I in S which contains mS.

Corollary. *Let (R, m) be a d-dimensional regular local ring. Let (S, n) be a d-dimensional regular local ring which birationally dominates R. If $S\neq R$, then $l(mS)<d$.*

Proof of the Corollary. Suppose $S\neq R$ and $l(mS)=d$. Say $x_1, \cdots, x_d$ is a minimal basis for m. Zariski's Main Theorem implies that $\operatorname{ht}(mS)<d$, so since S is regular, there is a relation

$$u_1x_1+\cdots+u_dx_d \in n^2,$$

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